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UKMT

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hanna Neumann

Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after Hanna Neumann (1914–1971).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Neumann_Hanna.html for more information, or search "Hanna Neumann mathematician" to find a wider range of sources.

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Version 1.1, Feb 2019

1. Set the scene for your mentoring work this year by discovering some facts about the life and work of Hanna Neumann.

Solution

See the cover sheet for a source of information. You may discover a connection between Hanna Neumann and UKMT.

2. A pair of twin primes consists of two successive odd integers p and $p + 2$ that are both primes. The integer between the two primes must, of course, be even. How many such integers are perfect squares? Justify your answer.

Solution

Suppose that the even integer between p and $p + 2$ is a perfect square. The square of an odd integer is odd, so our even integer must be the square of an even integer. (Are you sure that you can prove these last two statements?) It must therefore have the form $4n^2$, where $n \geqslant 1$ is a positive integer. Furthermore, $4n^2 - 1 = p$. Factorising yields $(2n - 1)(2n + 1) = p$. Since the only factors of p are 1 and p, this forces $2n - 1 = 1$. Therefore $n = 1$ and the even integer we have been considering is 4. The conclusion is that the only possible case is $p = 3$, $p + 1 = 2^2$, $p + 2 = 5$.

3. How many positive integers less than or equal to 1000 are not divisible by any of 2, 3 or 5?

Solution

It seems easier to think about divisibility than ‘not-divisibility’, so let’s try to count the number of positive integers less than or equal to 1000 that *are* divisible by at least one of 2, 3 or 5. We can then subtract this number from 1000 to get how many are not divisible by any of 2, 3 or 5.

There are

- 500 positive integers less than or equal to 1000 that are divisible by 2;
- 333 integers divisible by 3;
- 200 integers divisible by 5.

Now consider what will happen if we add up all these totals: can you see the snag? For instance, any integers divisible by both 2 and 3 will have been counted twice. The same is true for integers divisible by both 2 and 5 or by both 3 and 5. To take one example: $90 = 2 \times 3^2 \times 5$ is one such integer. Thus we must find out how many integers like this that there are and subtract their number from the total.

There are

- 166 integers divisible by both 2 and 3 (i.e. divisible by 6);
- 100 integers divisible by both 2 and 5 (i.e. divisible by 10); and
- 66 integers divisible by both 3 and 5 (i.e. divisible by 15).

We can take care of this problem by adding the first set of totals and subtracting the second set of totals. However, this introduces another difficulty: any number divisible by all three of 2, 3 and 5 (90 for example) will have been counted three times and then discounted again three times, resulting in it not being counted at all. Thus we must find out how many such integers there are and add them back to the total.

There are 33 integers divisible by all of 2, 3 and 5 (i.e. divisible by 30).

We conclude that there are $500+333+200-166-100-66+33 = 734$ positive integers less than or equal to 1000 that are divisible by at least one of 2, 3 or 5. Thus there are $1000 - 734 = 266$ positive integers less than or equal to 1000 that are not divisible by any of 2, 3 or 5.

Remark: The method of counting used here is called the inclusion-exclusion principle. You may want to look it up or discuss it with your mentor.

4. Will, Xavier, Yolanda and Zara are sixth-formers who like to work on maths problems together. Sometimes they tease their teacher by handing in a single, typed set of solutions. When questioned on one occasion, they make the following set of statements, each of which is either true or false.

Will:	It was Xavier.
Xavier:	It was Zara.
Yolanda:	It was not me.
Zara:	What Xavier says is wrong.

a) Let us suppose that exactly one of the four is lying and the other three are telling the truth. Who is lying? Who typed out the solutions?

b) Let us now suppose that exactly one of the four is telling the truth and the other three are lying. Who is telling the truth? Who typed out the solutions?

c) Now suppose that two of the statements are true and two are false. Who are the people who could have typed out the solutions?

d) Without prior knowledge of the number of true statements, what can we say about the identity of the typist?

Solution

Note first that the statements of Will and Xavier are contradictory, so one of them must be lying. The same is true of Xavier and Zara. If Xavier is lying, then Zara was not the typist and she is telling the truth about him. If Zara is lying, then what Xavier says is right.

a) One of Will and Xavier is lying. If Will is lying and Xavier is telling the truth, then Zara is lying. This is a contradiction as there is only one liar. Hence Will is telling the truth and Xavier is lying. Since Will is telling the truth, it was Xavier who typed out the solutions.

b) One of Xavier and Zara is telling the truth, and is the only truth-teller. Hence Will and Yolanda are lying and Yolanda is the typist. Consequently Xavier is lying and the truth-teller is Zara.

c) We know that precisely one of Xavier and Zara can be telling the truth and we examine the alternatives separately.
If the truth-teller is Xavier, then Zara typed out the solutions and hence Yolanda is also telling

the truth. The other liar is Will.
If the truth-teller is Zara, then Xavier is lying. Now the question is: which of Will and Yolanda is telling the truth? If Will is telling the truth in saying the typist was Xavier, then Yolanda is telling the truth in saying that it was not her. We now have three truth-tellers: Zara, Will and Yolanda. This is a contradiction and so Yolanda, not Will is telling the truth. Summing-up, Yolanda and Zara are telling the truth and Will and Xavier are lying. As the typist, we have now ruled out Xavier (Will is lying), Yolanda ("It was not me" is true) and Zara (Xavier is lying). Thus by elimination the typist is Will.

d) The various cases discussed have led to each of the students being revealed as the typist. Hence, in the absence of information about the number of true statements, any of the students could have typed out the solutions.

5. Prove that for all positive real numbers a, b, c:

a) $(a+b)^2 \geqslant 4ab$;

b) $\frac{a+b+c}{2} \geqslant \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}$.

Solution

It is often useful to work backwards when trying to prove an inequality by attempting to reduce it to something simpler which you know to be true. However you must always ensure that your logic works both ways and convince the reader of this as well. A good way of achieving this in your solution is to start with what you know to be true and derive the required inequality.

a) Multiplying out and simplifying we see that we wish to prove $a^2 + b^2 \geqslant 2ab$ which is equivalent to $(a-b)^2 \geqslant 0$ which is true as squares of real numbers are non-negative. In your solution you would start with “squares are non-negative so $(a-b)^2 \geqslant 0$” then multiply out, add $4ab$ to both sides, … to eventually get $(a+b)^2 \geqslant 4ab$.

b) The right-hand side of the inequality has three terms which we would rather not have to deal with all at once (putting them all onto a common denominator would be hard work!) so we might try and see if part *a*) will let us deal with each term separately.

We have $(a+b)^2 \geqslant 4ab$ and in order to get a $\frac{ab}{a+b}$ term we divide both sides by $a+b$ (note this is *positive* as a and b are) to get

$$a+b \geqslant \frac{4ab}{a+b},$$

and dividing by 4 we obtain,

$$\frac{a+b}{4} \geqslant \frac{ab}{a+b}.$$

Now we also have the analogous inequalities $\frac{b+c}{4} \geqslant \frac{bc}{b+c}$ and $\frac{c+a}{4} \geqslant \frac{ca}{c+a}$. Adding these all together gives

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \leqslant \frac{b+c}{4} + \frac{c+a}{4} + \frac{a+b}{4} = \frac{a+b+c}{2}.$$

6. An odd number of soldiers are stationed in a field, in such a way that all their pairwise distances from each other are distinct. (Note this means that, if you make a list of all the distances between each possible pair of soldiers, then all the distances in the list are distinct.) Each soldier is told to keep an eye on the soldier that is nearest to him. Prove that at least one soldier is not being watched.
Note: You may find it useful to start with three soldiers and then move on to five, before considering the general case.

SOLUTION

At the start of the solution it is a good idea to look at some cases involving a small number of soldiers. If there is just one soldier, there is no-one else to watch him and so one soldier is not being watched. If there are three soldiers, then there are three pairwise distances to consider; amongst these three there will be a smallest. Let the two soldiers separated by this smallest distance be A and B and let the third soldier be C. The soldiers A and B have to watch each other. Further, C watches A or B, depending on who is closer to him. There is no-one left to watch C and so again there is one soldier not being watched.

For convenience, denote the number of soldiers by n. In the previous case, where n was 3, it was immediately useful to focus on the two soldiers separated by the smallest distance in the list of distances. For subsequent values of n, there will always be such a pair and we shall continue to label them A and B. Moving on to $n = 5$, we see that there are now two cases that can occur involving the remaining three soldiers C, D and E, say. (You may find it useful to draw diagrams illustrating the two alternatives.)

- Case 1: A and B watch each other and one of them is watched by one of the three remaining soldiers.
- Case 2: neither A nor B is watched by any of the remaining three soldiers.

In Case 1, suppose C watches A or B. Then only D and E remain to watch C, D and E. Consequently at least one soldier remains unwatched. In Case 2 none of C, D and E is involved in watching either of the A, B pair. Hence C, D and E are covered by the argument given above for the case $n = 3$. Again there is one soldier not being watched.

Now notice that the preceding arguments are not special to the value $n = 5$. For Case 1, suppose that there are n soldiers altogether, where $n \geqslant 5$. Then, excluding A and B, there are $n - 2$ soldiers. We are assuming that one of these is watching one of A and B. There are therefore $n - 3$ soldiers left to watch the $n - 2$ soldiers besides A and B. Therefore there is one soldier not being watched.

In Case 2, the soldiers A and B are an isolated pair that can effectively be omitted from the rest of the solution. If we know that the result holds for a smaller odd number, then it holds for the set consisting of that smaller number, together with A and B. We used the fact that the result was true for $n = 3$ to prove it for $n = 5$. In the same way the $n = 5$ result can be used to clinch the argument for $n = 7, \ldots$ and so on. If you have met the method of proof by induction, then you will see it is just what it needed here. If you have not met induction, then you can stick with “and so on” but note Remark 1 below.

A Case 2 situation arose for the first time when $n = 5$ and we had already dealt with the base case ($n = 3$). The inductive hypothesis is that the result holds for a set of $2k - 1$ soldiers, where $k \geqslant 2$. Consider a set of $2k + 1$ soldiers. Then A and B are watching each other but can be split off from the remaining $2k - 1$ soldiers. The result holds for this smaller set and hence for the whole set of $2k + 1$ soldiers. The induction is complete.

Remark 1: proof by induction is a simple idea that formalises "and so on" arguments. There are many books and websites when you can look it up and you can consult your mentor if you would like to discuss it further.
Remark 2: assume that the solution refers to a previous era when soldiers could be referred to by male pronouns.

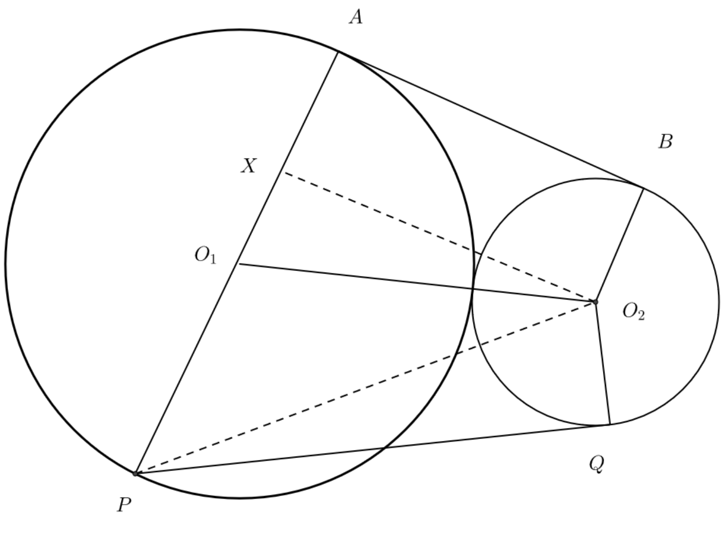
7. Two touching circles S and T share a common tangent which touches S at A and T at B. Let AP be a diameter of S and let a tangent to T touch T at Q and meet S at P. Show that $AP = PQ$.
[British Mathematical Olympiad Round 1 2006 Question 4]

Solution

In the diagram below O_1 is the centre of S and O_2 is the centre of T. This question is about circles and tangents so a good way to start is to marshal all you know about them.

First, since AB is a tangent and AP is a diameter, AB is perpendicular to AP. Relating this idea to the other circle, it might be useful to join B and O_2. Then BO_2 is also perpendicular to AB.

We now have a pair of parallel lines, each of which is perpendicular to AB. This configuration can be turned into a rectangle by dropping a perpendicular from O_2 onto AP. If the foot of this perpendicular is denoted by X, then AXO_2B is a rectangle. The diagram follows.



To simplify the working, let a be the radius of S and let b be the radius of T. Next recall that the line of centres O_1O_2 passes through the point of contact of the the two circles. It follows immediately that $O_1O_2 = a + b$. Also note that the perpendicular O_2X creates a right-angled triangle XO_1O_2. Since the question is about lengths, this suggests that Pythagoras' theorem is a tool to use. We have to consider PQ so, working down the diagram, it makes sense to join P and O_2. This produces another right-angled triangle, namely XPO_2. Again, joining O_2 and Q creates a third right-angled triangle in O_2PQ. The right angle is $\measuredangle PQO_2$, by another application of the radius-tangent property.

In the first right-angled triangle XO_1O_2, $XO_1 = a - b$. Note that the diagram implicitly assumes that $a > b$, so $a - b$ makes sense. We shall return to this point later. Applying Pythagoras' theorem yields

$$O_2X^2 = O_1O_2^2 - (a-b)^2 = (a+b)^2 - (a-b)^2 = 4ab.$$

It follows that, with another application of Pythagoras' theorem to triangle XPO_2,

$$O_2P^2 = O_2X^2 + XP^2 = 4ab + (a - b + a)^2 = 4ab + (2a - b)^2 = 4a^2 + b^2.$$

Finally, in triangle PQO_2, we have $PQ^2 = O_2P^2 - QO_2^2 = 4a^2 + b^2 - b^2 = 4a^2$.
Since we are dealing with lengths, this implies that $PQ = 2a$. However, AP, being a diameter of a circle with radius a, also has length $2a$. Hence the required result is proved.

The working above can serve for the case $b \geqslant a$ if $a - b$ is replaced by $|a - b|$.

8. *Something for you to investigate*

Twenty-first century primes: find all the primes between 2001 and 2100 inclusive, justifying your choice in each case.

Are any of your primes Mersenne numbers, of the form $2^n - 1$?

Are any of your primes Fermat numbers, of the form $2^{2^n} + 1$?

Have you found any twin primes, consecutive odd numbers which are both prime?

SOLUTION

- There are fourteen primes between 2001 and 2100 inclusive. They are

2003	2011	2017	2027	2029	2039	2053
2063	2069	2081	2083	2087	2089	2099

  They can be found by successively removing multiples of 2; then remaining multiples of 3; then remaining multiples of 5 and so on.

- The only power of 2 in the interval is $2^{11} = 2048$ but 2047 does not appear in the list.

- The list of Fermat numbers starts 3, 5, 17, 257, 65537, so no Fermat number appears in the list.

- There are three pairs of twin primes. Two pairs occur in a cluster of four primes between 2081 and 2089 inclusive.

- Are there more primes than you expected, or fewer? The Prime Number Theorem gives information on the distribution of primes. You may want to look it up or to ask your mentor about it.

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Hanna Neumann

Sheet 2

Solutions and comments

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1. Imagine that all the positive integers whose decimal representations include each of the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 exactly once are listed in ascending order of magnitude. The first digit is not allowed to be 0; thus the first three numbers in the list are

$$1023456789 \quad 1023456798 \quad 1023456879.$$

What is the millionth number in the list?

Solution

The first three numbers in the list are given: if we go on to consider all the relevant positive integers starting with 1, then we see that there are 9! altogether. It is thus a good idea to check on some larger, possibly unfamiliar, factorials. We shall need

$$7! = 5040 \quad 8! = 40320 \quad 9! = 362880.$$

Thus there are 362880 positive integers in the list that start with 1 and another 362880 that start with 2. Altogether there are $2 \times 362880 = 725760$ that start with 1 or 2. There are still $1000000 - 725760 = 274240$ left to deal with.

Next consider numbers in the list starting 30. (Make sure you understand why we do not consider all the numbers starting with 3.) The number of positive integers in the list starting with 30 is $8! = 40320$. Comparing this with 274240 suggests that we consider the numbers starting with 30, 31, 32, 34, 35 and 36. (Make sure you understand why we do not consider the numbers starting with 33.) The total tally here is $6 \times 40320 = 241920$. There are still $274240 - 241920 = 32320$ numbers to be dealt with.

The remaining calculations can be most clearly displayed in a table.

Factorial	Numbers starting	Tally	How many left?
$7! = 5040$	370, 371, 372, 374, 375, 376	$6 \times 7! = 30240$	$32320 - 30240 = 2080$
$6! = 720$	3780 and 3781	$2 \times 6! = 1440$	$2080 - 1440 = 640$
$5! = 120$	37820, 37821, 37824, 37825, 37826	$5 \times 5! = 600$	$640 - 600 = 40$
$4! = 24$	378290	$1 \times 4! = 24$	$40 - 24 = 16$
$3! = 6$	3782910 and 3782914	$2 \times 3! = 12$	$16 - 12 = 4$
$2! = 2$	37829150	$1 \times 2! = 2$	$4 - 2 = 2$

The remaining two relevant positive integers are 3782915406 and 3782915460. The second of these is therefore the millionth number in the list.

2. *a*) Prove that the positive integers which can be expressed as a difference of squares are exactly the odd integers and the multiples of 4.

b) Let k be a positive integer. Show that $4k(k + 1)^2$ can be expressed as $n^4 - m^2$ for some positive integers n, m.

Solution

a) First consider a multiple of 4, say $4k$, where $k \geqslant 1$. In breaking into this argument a possible first step is to look at some numerical cases. With their help, or by going directly to some simple algebra, we have $(k + 1)^2 - (k - 1)^2 = 4k$. This shows that any multiple of 4 can be expressed as the difference of two squares.

Turning to odd numbers, consider an odd integer $2k + 1$, where $k \geqslant 0$. The simple identity $2k + 1 = (k + 1)^2 - k^2$ shows that any odd integer can be also be expressed as the difference of two squares.

Is the solution complete? It is often a good idea to check that every word in the question has been used. What is the force of “exactly”? This requires us to be sure that the set of integers that can be expressed as the difference of two squares is exactly the same as the set consisting of the multiples of 4 and the odd integers. So far we have not excluded the possibility that there are integers, other than those in the second set, which can be expressed as the difference of two squares. It is time for the converse argument.

Let $N = n^2 - m^2$, where $n > m$, be a positive integer expressible as the difference of two squares. So far parity (whether an integer is odd or even) has been important, so let us bring that into the argument. If n and m are both even or both odd, then their sum and their difference are both even. Therefore, since $n^2 - m^2 = (n + m)(n - m)$, it follows that $n^2 - m^2$ is divisible by 4.

On the other hand, suppose that one of n and m is even and the other odd. In this case their sum and their difference are both odd. Consequently $(n + m)(n - m) = N$ is odd and this completes the converse argument.

b) This second part looks a lot worse but it turns out that nearly all the work has been done in the first part. There we saw that $4k = (k + 1)^2 - (k - 1)^2$. It follows that

$$4k(k + 1)^2 = [(k + 1)^2 - (k - 1)^2](k + 1)^2 = (k + 1)^4 - (k^2 - 1)^2.$$

Setting $k + 1 = n$ and $k^2 - 1 = m$ gives $4k(k + 1)^2 = n^4 - m^2$, as required.

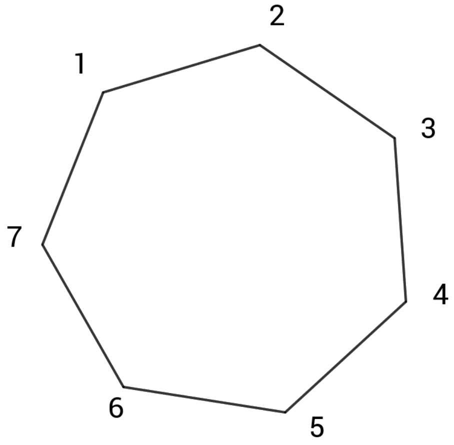
Remark: For example, $63 = 7 \times 9 = 3 \times 21$. *Following the method used in the proof,* $8^2 - 1^2 = 64 - 1 = 63$ *and also* $12^2 - 9^2 = 144 - 81 = 63$. *Further, the method works just as well for a prime. For example* $5 \times 1 = 3^2 - 2^2 = 9 - 4 = 5$.

3. The vertices of a regular heptagon (seven sides) are coloured either black or white. Prove that there are vertices of the same colour that form an isosceles triangle.

Can your argument be extended to any regular polygon with an odd number of sides? Justify your answer.

SOLUTION

Because the heptagon has an odd number of sides, there must be two adjacent vertices of the same colour. Assume these vertices are the ones labelled 2 and 3 in the diagram below. Assume further that, *without loss of generality,* these vertices are coloured black. (Note that the phrase in italics means that the subsequent argument would proceed in the same way if the two vertices happened to be coloured white. The only difference would be that the colours would be swapped.)



First suppose that either vertex 1 or vertex 4 is black. Then vertices 1, 2, 3 or vertices 2, 3, 4 form an isosceles triangle, all of whose vertices are black. If our supposition does not hold, then vertices 1 and 4 are both white. Now look at the vertex that is "opposite" those we have been considering, namely vertex 6. The trouble is that we do not know what colour it is; however, this turns out not to matter. If vertex 6 is black, then vertices 2, 3, 6 form a black isosceles triangle; if vertex 6 is white, then vertices 1, 4, 6 form a white isosceles triangle. The result is proved.

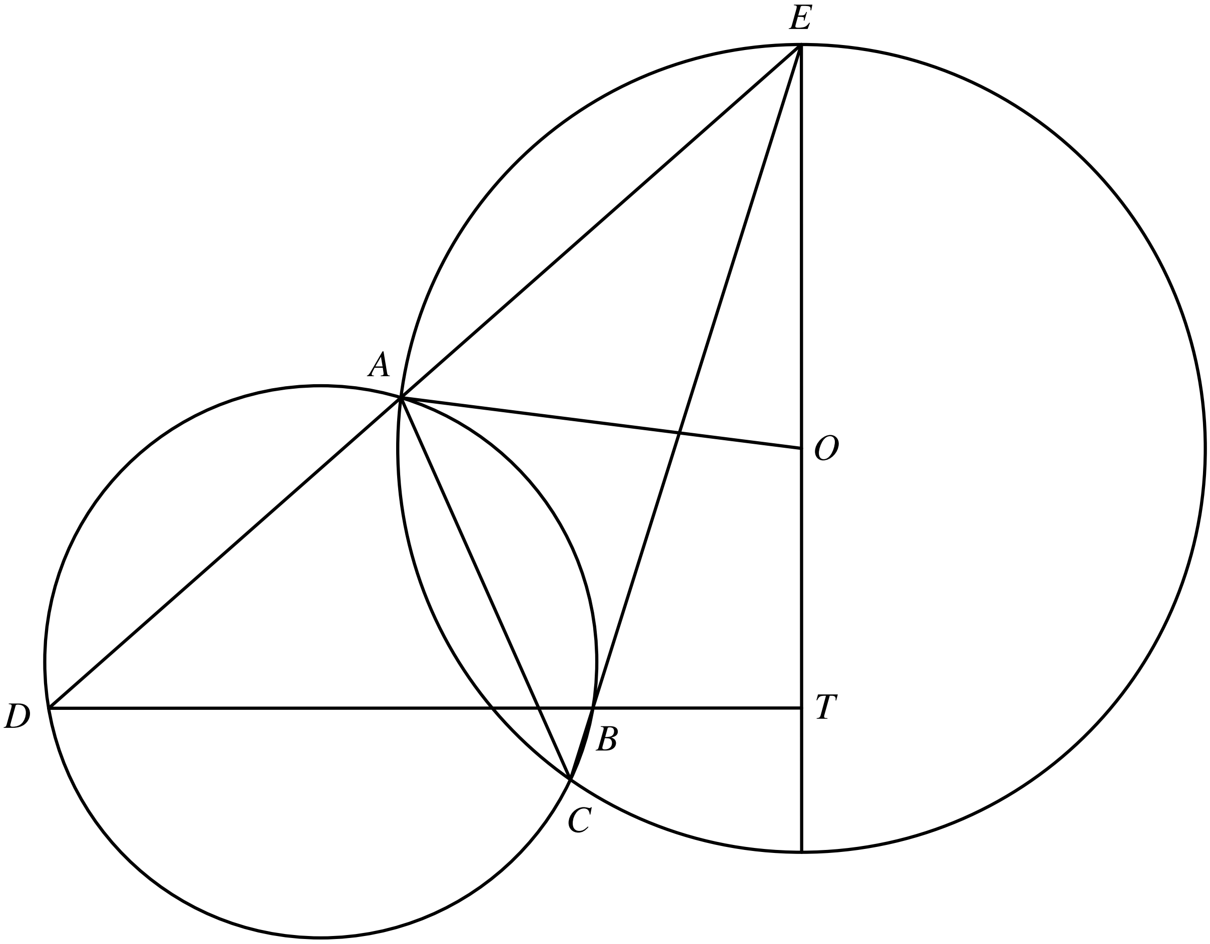
The foregoing proof can be adapted to any regular polygon with $2k + 1$ sides, where $k \geqslant 2$. Again assume that vertices 2 and 3 are coloured black. If either vertex 1 or vertex 4 is black, then the proof goes through as before. What happens now if vertices 1 and 4 are both white? Instead of counting on 3 vertices to vertex 6, count on k vertices to vertex $3 + k$. Note that counting on another k vertices from $3 + k$ takes us to the vertex with label $3 + k + k - (2k + 1)$, that is back to vertex 2. Consequently, as before, if vertex $3 + k$ is black, then vertices 2, 3, $3 + k$ form a black isosceles triangle. On the other hand, vertices 1, 4, $3 + k$ form a white isosceles triangle if vertex $3 + k$ happens to be white.

The remaining question is: what happens when $k = 1$? Then the polygon becomes an equilateral triangle. This will furnish an isosceles triangle with the desired property if all its vertices are the same colour. If, however, there are two vertices of one colour and one of the other, then no such isosceles triangle exists.

4. Points A, B, C and D all lie on a circle (in that order) with DA and CB produced meeting at point E. Let O be the centre of the circle through E, A and C. Prove that lines EO and BD are perpendicular.

SOLUTION

Let lines EO and BD meet at T, we are asked to show that $\measuredangle DTE = 90°$. There are a few possible strategies here: we are aiming to show that triangle DTE (or BTE, . . .) is right-angled and this could be done by calculating angles or by calculating lengths and using the converse of Pythagoras' theorem (make sure that you know what this says). There are many circle theorems giving relations between angles and a couple involving lengths, so we will try the angle approach first.



To show that $\triangle DTE$ is right-angled at T it is enough to show the other two angles of the triangle sum to 90° ($\measuredangle TED + \measuredangle EDT = 90°$) so we will try and relate these two.

As $ABCD$ is a cyclic quadrilateral we have (using 'angles in the same segment') $\measuredangle EDT = \measuredangle ADB = \measuredangle ACB = \measuredangle ACE$. $\measuredangle TED = \measuredangle OEA$ so we want to show $\measuredangle ACE + \measuredangle OEA = 90°$. O is the centre of circle EAC so (using 'angle at the centre is twice that at the circumference') $\measuredangle AOE = 2\measuredangle ACE$ hence we need to show $\frac{1}{2}\measuredangle AOE + \measuredangle OEA = 90°$.

Now triangle OEA is isosceles (OA and OE are radii) so $\measuredangle OEA = \measuredangle EAO$ and hence $180° = \measuredangle AOE + 2\measuredangle OEA$ which upon halving gives what we need.

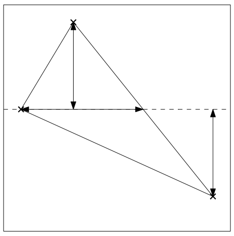
5. Let nine points be placed in a square box of side length 1, with no three points lying on the same line. Prove that there must be three points that form a triangle of area at most $\frac{1}{8}$.

SOLUTION

Divide the box into four squares of side length $\frac{1}{2}$. One of these smaller squares must contain at least 3 points: if each of the four squares contains only at most two points we can have at most eight points in total.

Remark: This kind of idea is sometimes called the Pigeonhole Principle. This states that if you *have $kn + 1$ pigeons contained in n pigeonholes then there must be some pigeonhole containing $k + 1$ pigeons (or more). In this case, the points are the pigeons and the small squares are the pigeonholes.*

Now take three points in the same $\frac{1}{2} \times \frac{1}{2}$ square. We want to show that the triangle formed by these three points has area $\leqslant \frac{1}{8}$. There are many ways to do this – see if you can come up with a different approach!



Order the three points by their distance from the bottom of the square. Add a line through the middle point that is parallel to the base of the square. This splits the triangle into two triangles with a common side, which must have length at most $\frac{1}{2}$. Use this side as the base in the $\frac{1}{2}\times$ base × height formula. Since the sum of the two heights is also at most $\frac{1}{2}$, this gives a total area of at most $\frac{1}{8}$.

6. Solve the equations

$$2x^2 + 3xy + 5y^2 = 3x^2 + 5xy + 2y^2 = 7.$$

Solution

We wish to solve

$$2x^2 + 3xy + 5y^2 = 3x^2 + 5xy + 2y^2 = 7. \tag{1}$$

While quadratic equations in one variable are quite familiar those with two are much less so. Fortunately we have been given three equations:

$$2x^2 + 3xy + 5y^2 = 7, \tag{2}$$
$$3x^2 + 5xy + 2y^2 = 7, \tag{3}$$
$$2x^2 + 3xy + 5y^2 = 3x^2 + 5xy + 2y^2. \tag{4}$$

It is not obvious what we can do with the first two equations at the moment but we can at least simplify the third to get

$$x^2 + 2xy = 3y^2. \tag{5}$$

We solve quadratic equations in one variable by completing the square (the quadratic formula is derived by doing this on a general quadratic) and this combines the squared and linear term into a square e.g. $x^2 + 4x = (x + 2)^2 - 4$. We can do the same thing to left-hand side of (5) to obtain $(x + y)^2 - y^2 = 3y^2$ and so

$$(x + y)^2 = (2y)^2.$$

We can now deduce that $x + y = 2y$ or $x + y = -2y$ so $x = y$ or $x = -3y$. We have exploited (4) fully but have not yet used (2) or (3) and we will do so now.

Case 1: $x = y$. Substituting into (2) we get $10y^2 = 7$ so $y = \pm\sqrt{\frac{7}{10}}$ and hence we have got two candidate solutions: $x = y = \sqrt{\frac{7}{10}}$ and $x = y = -\sqrt{\frac{7}{10}}$. We must check that these solutions satisfy (1), since our logic to obtain these candidate solutions may not be reversible. An alternative is to justify why our logic is reversible but checking is generally much easier to do. Both candidates work: in both cases $x^2 = xy = y^2 = \frac{7}{10}$ so $2x^2 + 3xy + 5y^2 = 10 \times \frac{7}{10} = 7$ and $3x^2 + 5xy + 2y^2 = 10 \times \frac{7}{10} = 7$.

Case 2: $x = -3y$. Substituting into (2) we get $14y^2 = 7$ so $y = \pm\sqrt{\frac{1}{2}}$ and hence we have got two candidate solutions: $x = -3\sqrt{\frac{1}{2}}$, $y = \sqrt{\frac{1}{2}}$ and $x = 3\sqrt{\frac{1}{2}}$, $y = -\sqrt{\frac{1}{2}}$. Both candidates work: in both cases $x^2 = \frac{9}{2}$, $xy = -\frac{3}{2}$, $y^2 = \frac{1}{2}$ so $2x^2 + 3xy + 5y^2 = 9 - \frac{9}{2} + \frac{5}{2} = 7$ and $3x^2 + 5xy + 2y^2 = \frac{27}{2} - \frac{15}{2} + 1 = 7$.

Thus there are four solutions which are $x = \sqrt{\frac{7}{10}}, y = \sqrt{\frac{7}{10}}$; $x = -\sqrt{\frac{7}{10}}, y = -\sqrt{\frac{7}{10}}$; $x = -3\sqrt{\frac{1}{2}}$, $y = \sqrt{\frac{1}{2}}$ and $x = 3\sqrt{\frac{1}{2}}, y = -\sqrt{\frac{1}{2}}$.

Remark: Another way to proceed from (5) *is to divide through by* y^2. *(Note that* y *cannot be zero: why not?) This division gives a quadratic in* $\frac{x}{y}$: $\left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right) = 3$. *This can be solved to give* $\frac{x}{y} = 1$ *or* -3 *so* $x = y$ *or* $x = -3y$ *as before. Dividing through by* y^2 *has the effect of converting an equation involving two variables* x *and* y *into one which involves only one variable,* $\frac{x}{y}$.

7. Find the value of

$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}.$$

[British Mathematical Olympiad Round 1 2007 Question 1]

Solution

Of course, this problem could be solved by direct calculation. However, with such huge numbers it would be difficult to be accurate. For a more subtle approach, a useful first step is to get rid of those long numbers by setting $2007 = a$. This has the additional benefit that algebra will tend to clarify the structure of the expression. We now have

$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2} = \frac{1^4 + a^4 + (1+a)^4}{1^2 + a^2 + (1+a)^2}.$$

Here the numerator is $2 + 4a + 6a^2 + 4a^3 + 2a^4 = 2(1 + 2a + 3a^2 + 2a^3 + a^4)$.
Also the denominator is $2 + 2a + 2a^2 = 2(1 + a + a^2)$.
On cancelling the 2's the desired expression becomes

$$\frac{1 + 2a + 3a^2 + 2a^3 + a^4}{1 + a + a^2}.$$

It is inviting to think that the denominator is a factor of the numerator and you may recognise the numerator as a square. If this is not obvious, then you can test the invitation by setting

$$1 + 2a + 3a^2 + 2a^3 + a^4 = (1 + a + a^2)(1 + ka + a^2).$$

On multiplying out, the right-hand side becomes

$$1 + a + a^2 + ka + ka^2 + ka^3 + a^2 + a^3 + a^4 = 1 + (k+1)a + (k+2)a^2 + (k+1)a^3 + a^4.$$

Comparing coefficients with those in the numerator it is clear that $k = 1$ and hence that the numerator is the square of the denominator. This in turn implies that the whole expression is equal to $1 + a + a^2$. Returning to the numbers shows that the value of the required expression is

$$1 + 2007 + 2007^2 = 2008 + 4000000 + 2 \times 2000 \times 7 + 49.$$

(Note that the square can be calculated inline by writing $2007^2 = (2000 + 7)^2$.) Doing the arithmetic shows that

$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2} = 4030057.$$

8. *Something for you to investigate*

Solving cubic equations: the method considered here is usually known as Cardano's method, after it was published by Girolamo Cardano (1501 - 1576). However, it is now generally agreed that the first person to devise a method of solution for a cubic equation was Scipione del Ferro (1465 - 1526). The following instructions show how the method works for a cubic equation of the form $x^3 + cx = d$. During the Renaissance, European mathematicians often used rhymes as mnemonics; a rough translation of one is given, with extra explanations.

When the cube and the things together	
Are equal to some discrete number,	*Equation is* $x^3 + cx = d$.
Find two other numbers differing in this one.	*Find* u, v *with* $u - v = d$.
Then you will keep this as a habit	
That their product will always be equal	
Exactly to the cube of a third of the things.	*Also* $uv = (c/3)^3$.
The remainder then as a general rule	
Of their cube roots subtracted	
Will be equal to your principal thing.	*Then* $x = \sqrt[3]{u} - \sqrt[3]{v}$.

Check that Cardano's method does provide a solution of the equation.

Spot a solution of $x^3 + 6x = 20$.

Use Cardano's method to obtain a solution of $x^3 + 6x = 20$.

Reconcile the two solutions you have obtained.

Solution

- Check that substituting $\sqrt[3]{u} - \sqrt[3]{v}$ for x in the left-hand side of the equation does give d on the right -hand side.
- By inspection $x = 2$ is a solution.
- Cardano's method yields the solution $x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$.
- The solution above may be written as $x = \sqrt[3]{6\sqrt{3} + 10} - \sqrt[3]{6\sqrt{3} - 10}$ but it is not clear that this is equal to 2. However, it is possible that, for example, $6\sqrt{3} + 10$ is the cube

of an expression of the form $a + b\sqrt{3}$. You can verify this by expanding $(a + b\sqrt{3})^3$ and comparing it with $6\sqrt{3} + 10$. It is then possible to see that $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$ simplifies to 2.

United Kingdom
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Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hanna Neumann

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after Hanna Neumann (1914–1971).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Neumann_Hanna.html for more information, or search “Hanna Neumann mathematician” to find a wider range of sources.

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Version 1.3, Nov 2020

1. The prime p and the integers x and y are such that $4xy = p(p - 2x - 2y)$. What is the largest value of $p^2 + x^2 + y^2$?

Solution

The variables in the question are integers and primes, so it seems sensible to consider divisibility. The left-hand side of the equation is even and so $p(p - 2x - 2y)$ is also even. As p is prime, it is either equal to 2 or it is odd. If p is odd, then both p and $p - 2x - 2y$ are odd, making the right-hand side of the equation odd. This is impossible and so $p = 2$.

Substituting into the equation gives $4xy = 2(2 - 2x - 2y)$, which we can rearrange into the form $xy + x + y - 1 = 0$. It would be helpful to pull all the x's and y's into a single term. We can do this by the following useful trick. We can write $xy + x + y + 1 = 2$, which factorises as $(x + 1)(y + 1) = 2$. Now x and y are integers, so $x + 1$ and $y + 1$ are two integers whose product is 2. The only possibilities are 2 and 1 or -2 and -1. It follows that x and y must be 1 and 0 or -3 and -2. These values give $p^2 + x^2 + y^2 = 5$ or 17. Thus the largest value of $p^2 + x^2 + y^2$ is 17, occurring when $p = 2$, $x = -3$ and $y = -2$ and when $p = 2$, $x = -2$ and $y = -3$.

2. The last International Mathematical Olympiad to be held in the UK took place in Bath in 2019. I am thinking of an anniversary (which is nothing to do with mathematics) occurring between that year and 2099 inclusive. The anniversary year is the product of a prime, the square of another prime and the cube of a third prime (all three primes are distinct). Furthermore, the sum of the digit sums* of the cube, the square and the prime is 10 more than the digit sum of the anniversary year. What is the anniversary year?
When you have found the anniversary year, you may want to identify the event that it commemorates.
**The digit sum of an integer is the sum of its digits.*

Solution

The anniversary year can be expressed as p^3q^2r, where p, q and r are distinct primes. We need to get some idea of how big these primes can be and a sensible start is to consider the cube first, since cubes increase most quickly. Now $11^3 = 1331$ and the smallest possible multiplier for this number is 2. However, $2 \times 1331 = 2662$, which is out of range. Therefore the cube must be one of

$$7^3 = 343 \quad 5^3 = 125 \quad 3^3 = 27 \quad 2^3 = 8.$$

The cube has to be multiplied by a square and 343×3^2 is greater than 2099. However, $343 \times 2^2 = 1372$ is not. Nevertheless, this can be discounted because the smallest possible prime multiplier left is 3 (note p, q and r have to be distinct) and $1372 \times 3 > 2099$.

We could proceed in the same way and multiply the remaining cubes by the possible squares but this is beginning to look laborious. Shifting the viewpoint, look for the multiples of the relevant cubes that lie in the required range.

Since $16 \times 125 = 2000$ and $17 \times 125 = 2125$, 5^3 does not contribute any candidates. Moving on to 3^3, we have

$$75 \times 27 = 2025 \quad 76 \times 27 = 2052 \quad 77 \times 27 = 2079,$$

while 74×27 is too small and 78×27 is too large.

The first of these candidates can be discounted because $75 \times 27 = 3 \times 5^2 \times 3^3$, which contains a repeated prime. Likewise the third is not a candidate because 77 does not have a square factor. However, $2052 = 3^3 \times 76 = 3^3 \times 2^2 \times 19$ has the right pattern of factors. Considering the digit sums: the digit sum of 2052 itself is 9; the sum of the digit sums of the cube, square and prime is $2 + 7 + 4 + 1 + 9 = 23$. Thus the digit sum condition is not satisfied.

Finally, the multiples of $2^3 = 8$ lying between 2019 and 2099 range from $253 \times 8 = 2024$ to $262 \times 8 = 2096$. We can immediately discount all the even multipliers because they repeat the prime 2. This leaves 253, 255, 257, 259 and 261. The prime factorisations are

$$253 = 11 \times 23 \quad 255 = 3 \times 5 \times 17 \quad 257 \quad 259 = 7 \times 37 \quad 261 = 3^2 \times 29.$$

Thus we have one product of the required form, namely $2^3 \times 3^2 \times 29 = 2088$. Considering the digit sums: the digit sum of 2088 itself is 18; the sum of the digit sums of the cube, square and prime is $8 + 9 + 2 + 9 = 28$. This time the digit sum condition is satisfied and we conclude that the anniversary year is 2088.

The history: the year 2088 will be the 500th anniversary of the defeat of the Spanish Armada in 1588.

3. *a*) The *absolute value* (sometimes called the *modulus*) $|u|$ of the real number u is its numerical value when its sign is ignored. Thus, for example, $|6.25| = 6.25$, $|0| = 0$ and $|-\pi| = \pi$. Explain why, for any two real numbers a and b,

$$a + b - |a - b| = 2 \times \min(a, b).$$

Remark: Here $\min(a, b)$ *is the smaller of the numbers a and b. If a = b, then* $\min(a, b)$ *is their common value.*

b) Suppose that the positive integer n is odd. Victoria writes the sum

$$S = 1 + 2 + 3 + \ldots + (2n - 1) + 2n$$

on the board. Then she picks any two of the integers, a and b say, and removes them from the sum. She replaces them in the sum by $|a - b|$. She repeats this operation until just one integer remains. Prove that this integer must be odd.

Solution

a) If $a \geqslant b$, then $a - b \geqslant 0$ and so $|a - b| = a - b$. It follows that

$$a + b - |a - b| = a + b - (a - b) = 2b = 2 \times \min(a, b).$$

On the other hand, if $a < b$, then $a - b < 0$ and so $|a - b| = b - a$. This time

$$a + b - |a - b| = a + b - (b - a) = 2a = 2 \times \min(a, b).$$

b) First, $S = \frac{1}{2} \times 2n(2n + 1) = n(2n + 1)$. This is an odd number because n is odd (given) and $(2n + 1)$ is odd. Furthermore, every time we remove terms a and b from S and replace them by $|a - b|$, we are subtracting $a + b - |a - b|$. Thus by (a) above we are subtracting an even number from S and so the sum is still odd. If we repeat the process until the sum reduces to a

single term, then this remaining integer will be odd.
Remark: This is an example of an argument involving an invariant*, that is, something that does not change. In this case the invariant is the parity of the sum: it remains odd.*

4. A positive integer $a > 1$ is given in decimal notation. We copy it twice and obtain an integer b, which happens to be a multiple of a^2. Find all the possible values of $\frac{b}{a^2}$.
(*Note: For example, if* $a = 365$*, then* $b = 365365$.)

Solution

Suppose that a is an n-digit integer. It has a decimal representation of the form

$$a = k_{n-1} \times 10^{n-1} + k_{n-2} \times 10^{n-2} + \ldots + k_1 \times 10 + k_0.$$

When the integer b is constructed, it is given by $b = a(10^n + 1)$. (Ensure that you understand why this is so.) Thus

$$\frac{b}{a^2} = \frac{10^n + 1}{a} = d, \text{ say, where } d \text{ is an integer.}$$

Since a has n digits, we must have $2 \leqslant d \leqslant 10$. (Ask yourself: why is $d \neq 1$?) First note that ad is equal to the odd integer $10^n + 1$ and so d cannot be even. Similarly, 3 and 5 are not factors of $10^n + 1$ and so d cannot be equal to any of 3, 5, 9 or 10. This leaves 7 as the only possible value for d.

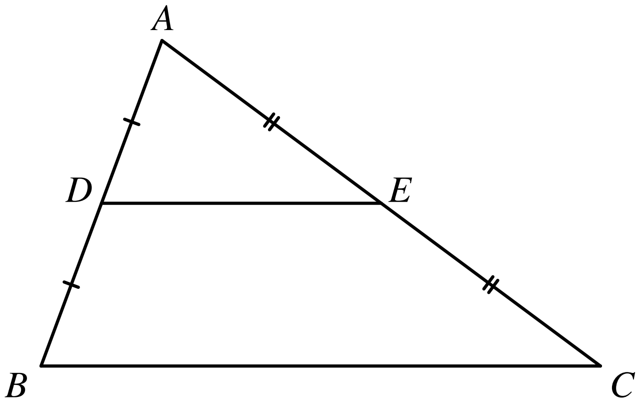
Note that neither 11 nor 101 is a multiple of 7 but $1001 = 7 \times 143$. Hence the conditions of the question are satisfied when $a = 143$ and $b = 143143$.

5. *a*) Let ABC be a triangle, D the midpoint of AB, E the midpoint of AC. Prove that DE is parallel to BC with $DE = \frac{1}{2}BC$

b) Let A, B, C and D be four points in the plane no three of which are collinear (lie on a line). Let K, L, M, N, P and Q be the midpoints of AB, BC, CD, DA, AC and BD respectively. Prove that lines KM, LN and PQ are concurrent (meet at a single point).

Solution

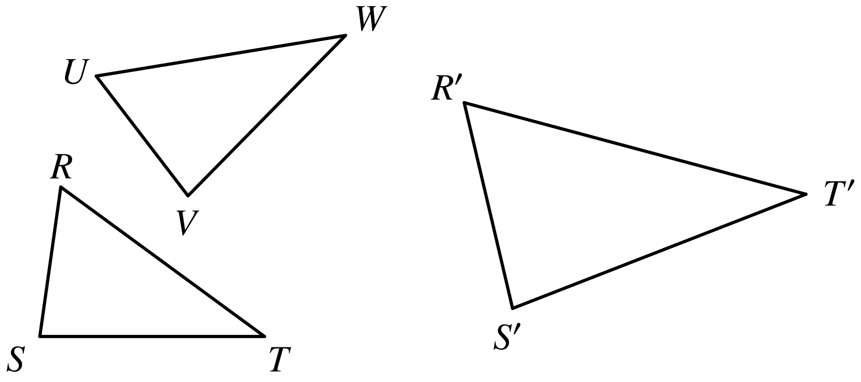
a) This is the *midpoint theorem.* We have the following simple diagram.



Triangles ADE and ABC look similar and in fact that is precisely what we want to show: if we could show this then $\measuredangle ABC = \measuredangle ADE$ so DE and BC are parallel and furthermore we would have $\frac{DE}{BC} = \frac{AD}{AB} = \frac{1}{2}$.

Many conditions for similarity are very similar to those for congruence:

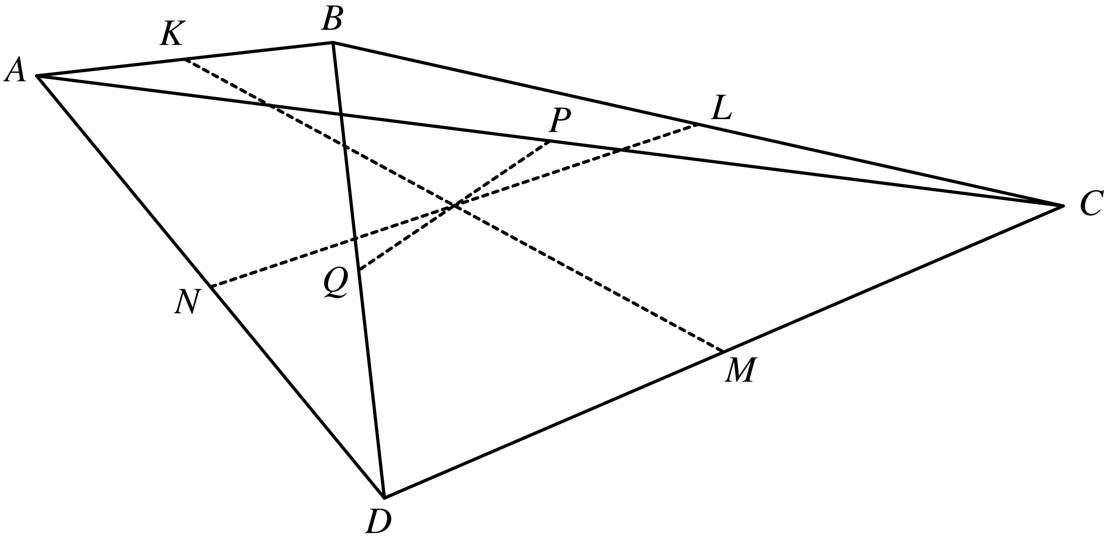
Acronym	Congruence	Similarity
SSS	All three sides equal: $RS = UV, ST = VW, TR = WU$	All three sides in ratio: $\frac{R'S'}{RS} = \frac{S'T'}{ST} = \frac{T'R'}{TR}$
SAS	Two sides and included angle equal: $RS = UV, ST = VW,$ $\measuredangle RST = \measuredangle UVW$	Two sides in ratio and included angle equal: $\frac{R'S'}{RS} = \frac{S'T'}{ST},$ $\measuredangle RST = \measuredangle R'S'T'$
AAS or ASA	Two angles and one side equal: $\measuredangle RST = \measuredangle UVW,$ $\measuredangle STR = \measuredangle VWU, RS = UV$	Two angles equal: $\measuredangle RST = \measuredangle R'S'T',$ $\measuredangle STR = \measuredangle S'T'R'$



For the problem at hand the SAS condition is perfect: $\measuredangle EAD = \measuredangle CAB$ and $\frac{AD}{AB} = \frac{1}{2} = \frac{AE}{AC}$ so $\triangle ADE \sim \triangle ABC$ as required.

Remark: Another way of doing this is thinking of the enlargement centre A scale factor 2: this sends D to B and E to C so $DE = \frac{1}{2}BC$ *and as enlargements preserve directions lines DE and BC are parallel.*

b) There are many midpoints so it makes sense to attempt to use part a)!



We are trying to prove that all three lines KM, LN and PQ are concurrent. We could do this by letting the intersection of KM and LN be a point X and showing that PXQ is a line or by finding some point that all three lines pass through. Anyway it makes sense to focus initially on only two of the lines, say KM and LN, to simplify what is going on.

K is the midpoint of AB and L is the midpoint of BC so, by part a, $KL \parallel AC$ and $KL = \frac{1}{2}AC$. Similarly $LM \parallel BD$, $LM = \frac{1}{2}BD$, $MN \parallel AC$, $MN = \frac{1}{2}AC$, $NK \parallel BD$, $NK = \frac{1}{2}BD$. Putting these together gives $KL \parallel MN$, $LM \parallel NK$ (and $KL = MN$, $LM = NK$) so that $KLMN$ is a parallelogram. We are interested in where the diagonals KM and LN of the parallelogram meet.

Now the diagonals of a parallelogram bisect each other (try to prove this using congruent triangles) so KM and LN meet at the midpoint of KM (which is also the midpoint of LN).

We would like to show that PQ also passes through this point. By the same reasoning as above $KPMQ$ is a parallelogram so PQ passes through the midpoint of KM so we are done: KM, LN and PQ all pass through the midpoint of KM so are concurrent.

6. *a*) Prove that for any cuboid the square of the length of its long diagonal is at least half of its surface area.

b) Prove that for any positive real numbers a, b and c:

$$ab + bc + ca \geqslant a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

In each part state when equality occurs.

Solution

a) Let the length of the sides of the cuboid be x, y and z so its longest diagonal has length $\sqrt{x^2 + y^2 + z^2}$, it surface area is $2(xy + yz + zx)$. We are asked to prove that

$$x^2 + y^2 + z^2 \geqslant xy + yz + zx. \qquad (1)$$

You might recall the first part of Question 5 on sheet 1 where we used the inequality

$0 \leqslant (a-b)^2 = a^2 + b^2 - 2ab$ which relates sums of squares to products. The inequality at hand is very similar except we now have three variables.

In the second part of Question 5 on sheet 1 we split up our inequality into three similar parts and that works well here to: we have $x^2 + y^2 \geqslant 2xy$, $y^2 + z^2 \geqslant 2yz$ and $z^2 + x^2 \geqslant 2zx$. Summing these gives

$$2x^2 + 2y^2 + 2z^2 \geqslant 2xy + 2yz + 2zx,$$

and dividing by two gives what we want.

b) We try and use part *a*) and we do this by setting $x = \sqrt{ab}$, $y = \sqrt{bc}$ and $z = \sqrt{ca}$ in (1) as then $x^2 = ab$, $y^2 = bc$ and $z^2 = ca$ so the left-hand sides match. Furthermore the right-hand sides do too so we have what we want.

Now in order to have equality in part *a*) we need equality in each of our three inequalities $x^2 + y^2 \geqslant 2xy$, $y^2 + z^2 \geqslant 2yz$ and $z^2 + x^2 \geqslant 2zx$. Those inequalities come from the squares $(x-y)^2, (y-z)^2, (z-x)^2$ all being non-negative. Thus we have equality only if each of the squares is zero, that is only if $x = y = z$. So to have equality the shape must be a cube and in fact if it is a cube then we do indeed get equality.

For part *b*), we need equality in (1) which we have just said occurs if and only if $x = y = z$ so we have equality in part *b*) if and only if $\sqrt{ab} = \sqrt{bc} = \sqrt{ca}$ i.e. we need $a = b = c$. If this occurs then both sides are $3a^2$ so we do get equality.

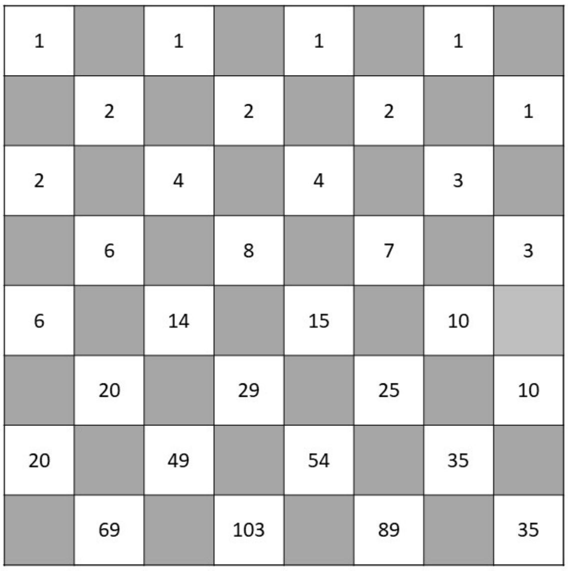
7. Consider a standard 8 × 8 chessboard consisting of 64 small squares coloured in the usual pattern, so 32 are black and 32 are white. A zig-zag path* across the board is a collection of white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

[British Mathematical Olympiad Round 1 2008 Question 1]

**This BMO1 question assumed that the zig-zag paths crossed the board from top to bottom, not from side to side as well.*

Solution

We can calculate the number of paths to each white square, row-by-row. In the first row, each white square can be reached in only way. After that, each white square can be reached from the one or two white squares diagonally above it: the number of ways it can be reached is the sum of the numbers in these one or two white squares. Repeated calculations produce the following diagram.



1	▨	1	▨	1	▨	1	▨
▨	2	▨	2	▨	2	▨	1
2	▨	4	▨	4	▨	3	▨
▨	6	▨	8	▨	7	▨	3
6	▨	14	▨	15	▨	10	░
▨	20	▨	29	▨	25	▨	10
20	▨	49	▨	54	▨	35	▨
▨	69	▨	103	▨	89	▨	35

As can be seen, there are respectively 35, 89, 103 and 69 ways of reaching the four white squares in the last row. Thus, in total, there are 35 + 89 + 103 + 69 = 296 zig-zag paths.

8. Is it possible to list the non-negative integers as $n_1, n_2, n_3, \ldots$ (with each non-negative integer appearing precisely once) in such a way that, for each $i = 1, 2, 3, \ldots$, we have $n_{i+1} = n_i + 2$ or $n_{i+1} = n_i - 2$ or $n_{i+1} = n_i \times 2$ or $n_{i+1} = n_i/2$?

Solution

It *is* possible to list the non-negative integers in the way described but what should the starting-point be? The only way to reach 0 is by subtracting 2 and the only way to leave 0 is by adding 2. It is impossible to do both these things without visiting 2 twice and so the list must start 0, 2. If the next number were not 1, then a similar problem would arise with visiting 1. The first four numbers must therefore be 0, 2, 1, 3.

Now it is not obvious whether to put 5 or 6 next and in fact both will work. We need to discover a reliable way to extend an initial fragment to a longer one, in such a way that the resulting sequence satisfies the conditions of the problem. Here is such a promising continuation.

$$0 \quad 2 \quad 1 \quad 3 \quad 6 \quad 4 \quad 8 \quad 10 \quad 5 \quad 7$$

Further, we can see that leads to a way of listing all the numbers up to 11:

$$0 \quad 2 \quad 1 \quad 3 \quad 6 \quad 4 \quad 8 \quad 10 \quad 5 \quad 7 \quad 9 \quad 11.$$

Is it possible to continue this further? Yes it is and one way to continue is this:

$$\ldots \quad 11 \quad 22 \quad 20 \quad 18 \quad 16 \quad 14 \quad 12 \quad 24 \quad 26 \quad 13 \quad 15 \quad 17 \quad 19 \quad 21 \quad 23 \quad 25 \quad 27,$$

which gives a list of all the numbers up to 27.

Having found this example, perhaps we can see that the method used works more generally: given a legal list of all the numbers up to some odd number $2n + 1$ (and ending with $2n + 1$ itself), this can be extended to a list of all the numbers up to $4n + 7$ by continuing as follows. First obtain

$$\ldots \quad 2n+1 \quad 4n+2 \quad 4n \quad 4n-2 \quad \ldots \quad 2n+2.$$

Here you double $2n + 1$ and then step down by repeatedly subtracting 2 through all the even numbers larger than $2n + 1$ until you reach $2n + 2$. Then you double $2n + 2$ to get $4n + 4$, add 2 to get $4n + 6$, then halve to get $2n + 3$, then step up by repeatedly adding 2. Continue through all subsequent odd numbers until $4n + 7$ is reached. Writing this out, we obtain

$$4n+4 \quad 4n+6 \quad 2n+3 \quad 2n+5 \quad \ldots \quad 4n+7.$$

The idea is therefore to keep subtracting 2 many times to visit a run of even integers and later to add 2 many times to visit the odd numbers which were skipped on the way down. We have to fiddle about to get started, to join up these two long runs and to end up at a large odd number.

This proves that it is possible to list all the non-negative integers, by starting with 0, 2, 1, 3 and then repeatedly extending this initial run in the way described.

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Hanna Neumann

Sheet 4

Solutions and comments

This programme of the Mentoring Scheme is named after Hanna Neumann (1914–1971).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Neumann_Hanna.html for more information, or search "Hanna Neumann mathematician" to find a wider range of sources.

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1. *a*) How many ways are there to choose two distinct numbers from the first n positive integers? Check your answer by writing out all of the possibilities when $n = 4$.

b) How many ways are there to choose two distinct numbers from the first n positive integers, where the two numbers chosen are not consecutive?

c) Let d be a non-negative integer. How many ways are there to choose two distinct numbers from the first n positive integers, where the difference between the two numbers chosen is at least d?

Solution

a) Suppose we choose first one integer and then another. There are n possible choices for the first integer. There are then $n - 1$ possible choices for the second integer, which must be different from the first. However, we have counted each pair of integers twice: once for each order they could be in. Thus the total number of possible distinct pairs is

$$\frac{n(n-1)}{2}.$$

As a check, let's consider the case when $n = 4$. There are indeed $n(n-1)/2 = 6$ possible distinct pairs: $(1,2), (1,3), (1,4), (2,3), (2,4)$ and $(3,4)$.

b) The possible pairs of consecutive integers are $(1,2), (2,3), \dots, (n-1, n)$ and so there are $n - 1$ such pairs. Thus the number of pairs of distinct integers which are *not* consecutive is

$$\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}.$$

c) Let $f(n, d)$ be the numbers of ways to choose two distinct numbers of difference at least d from the first n positive integers. Note that from parts *(a)* and *(b)* we have that $f(n, 1) = n(n-1)/2$ and $f(n, 2) = (n-1)(n-2)/2$. Let's consider the case $d = 3$ to see if we can spot a pattern.

The possible pairs of integers with difference equal to 2 are $(1,3), (2,4), \dots, (n-2, n)$ and so there are $n - 2$ such pairs. Thus the number of pairs of distinct integers which are not consecutive *nor* have difference 2 is

$$f(n,3) = f(n,2) - (n-2) = \frac{(n-1)(n-2)}{2} - (n-2) = \frac{(n-2)(n-3)}{2}.$$

From these calculations we might guess that general formula is

$$f(n,d) = \frac{(n-d+1)(n-d)}{2}.$$

There are (at least) two possible ways we might prove this.

1) By induction: since the method of induction has been discussed on previous sheets, a formal induction argument will be given here. When $d = 1$, substituting in the formula gives $f(n,1) = \frac{(n-1+1)(n-1)}{2} = \frac{n(n-1)}{2}$. But this agrees with the result in part *a)*, so the base case is proved.

   Suppose that $f(n,d) = \frac{(n-d+1)(n-d)}{2}$ holds for some d. We want to show that it also holds for $d + 1$.

To get from $f(n,d)$ to $f(n,d+1)$, we must subtract the number of pairs of integers with difference exactly d. The possible pairs of integers with difference equal to d are $(1,d+1),(2,d+2),\dots,(n-d,n)$ and so there are $n-d$ such pairs. Thus, as required, we have:

$$f(n,d+1)=f(n,d)-(n-d)=\frac{(n-d+1)(n-d)}{2}-(n-d)=\frac{(n-d)(n-d-1)}{2}.$$

2) Combinatorially, that is by a counting method.

Note that $\frac{(n-d+1)(n-d)}{2}$ is the number of ways of choosing two distinct integers from $\{1,2,\dots,n-d+1\}$. If we take such a pair and add $d-1$ to the larger of them, we get a pair of integers in $\{1,2,\dots,n\}$ such that the difference between them is at least d. (In understanding this step it may help to consider integers N and $N+1$ in the relevant range.) Note that this is reversible: given a pair of integers in $\{1,2,\dots,n\}$ with difference at least d, subtracting $d-1$ from the larger gives a pair of distinct integers in $\{1,2,\dots,n-d+1\}$.

Thus the number of pairs of integers in $\{1,2,\dots,n\}$ with difference at least d must be the same as the number of pairs of distinct integers in $\{1,2,\dots,n-d+1\}$. Equivalently,

$$f(n,d+1)=\frac{(n-d)(n-d-1)}{2}.$$

2. *Let a and b be integers and n a positive integer. We write* $a\equiv b \pmod n$ (say "a is congruent to b modulo n") *if the difference* $a-b$ *is divisible by* n. *Congruence fits in nicely with the operations of arithmetic and three examples of this are explored in parts a) and b) of the question.*

In what follows let a, b, c, d be integers and let n be a positive integer such that $a\equiv c \pmod n$ and $b\equiv d \pmod n$.

a) Show that $a-b\equiv c-d \pmod n$.

b) Show that $ab\equiv cd \pmod n$. Show further that, if k is a positive integer, then $a^k\equiv c^k \pmod n$.

c) Show that 41 divides $2^{20}-1$.

SOLUTION

a) The conditions given imply that there are integers k_1 and k_2 such that $a-c=k_1n$ and $b-d=k_2n$. Hence $(a-b)-(c-d)=(a-c)-(b-d)=(k_1-k_2)n$. But this is the required result that $a-b\equiv c-d \pmod n$.

b) Again we have

$$ab-cd=ab-ad+ad-cd=a(b-d)+(a-c)d=ak_2n+k_1nd=(ak_2+dk_1)n.$$

This is the first result $ab\equiv cd \pmod n$. Setting $b=a$ and $d=c$ yields $a^2\equiv c^2 \pmod n$. Repeating the argument gives the second result for any positive value of k; this method could also be formalised by the use of induction.

c) Here is a question for which direct calculation could be used since $2^{20} = \left(2^{10}\right)^2$ and $2^{10} = 1024$. However, such a method would be laborious and offer lots of opportunities for making slips. The calculations can be simplified, and hence speeded up, by using arithmetic modulo 41.

Exploring this idea, we might look for powers of 2 that are close to multiples of 41. The first likely candidate is $3 \times 41 = 123$, which is close to $2^7 = 128$. Since $128 - 5 = 3 \times 41$, we have $2^7 \equiv 5 \pmod{41}$. Then, using the result on powers in part *b)*, it follows that $2^{21} \equiv \left(2^7\right)^3 \equiv 5^3 \equiv 125 \equiv 2 \pmod{41}$. (Make sure you understand why the last step is valid.) We now know that there is an integer k such that $2^{21} - 2 = 2(2^{20} - 1) = 41k$. Since 41 is odd, this relationship also implies that k is even, say $k = 2m$. Cancelling the factors of 2 yields $2^{20} - 1 = 41m$, which states that 41 divides $2^{20} - 1$.

On reviewing the previous work, it may occur to you that the modular arithmetic circumvented a lot of calculation but its effectiveness petered out at the end of the argument. It would be an improvement to work with a power 2^a, where a was a factor of 20. Using the previous notion of searching for suitable powers of 2 that are fairly close to multiples of 41, we need go no further than noticing that $2^5 + 9 = 32 + 9 = 41$. Initially, this does not look promising as we have not reached the smallest multiple of 41. However, the sum can be rewritten as $32 - (-9) = 41$. If we include negative integers in the argument, then this converts to $2^5 \equiv -9 \pmod{41}$. Using the power rule again yields

$$2^{20} - 1 \equiv \left(2^5\right)^4 - 1 \equiv (-9)^4 - 1 \equiv 81^2 - 1 \pmod{41}.$$

Using the same idea again, note that $81 \equiv -1 \pmod{41}$. The final step in the calculation is

$$2^{20} - 1 \equiv 81^2 - 1 \equiv (-1)^2 - 1 \equiv 0 \pmod{41}.$$

(Check that you understand why this line is equivalent to the statement that 41 divides $2^{20} - 1$.)

3. For each of the following equations find all pairs of non-negative integers (m, n) satisfying them:

a) $2^m + 9 = n^2$;

b) $2^m + 5 = n^2$.

Solution

This question is a mixture of algebra and number theory: algebra since we are solving equations and number theory since we are dealing with integers only. Thus we can use ideas of divisibility and modular arithmetic. In the previous question we saw how modular arithmetic can simplify calculations where we care only about divisibility and remainders rather than the numerical answer (we did not mind what $2^{20} - 1$ is, just that 41 divided it). Here we see how it can simplify integer equations.

a) We wish to solve $2^m + 9 = n^2$. One thing to notice is that if we pull the 9 across to the other side we obtain a difference of two squares: $2^m = (n-3)(n+3)$. $n-3$ and $n+3$ are multiplying together to get a power of 2 so must themselves be powers of 2 (do you see why?).

That is $n - 3 = 2^a$ and $n + 3 = 2^b$ where a and b are non-negative integers with $a < b$. To get rid of n we can subtract these to get

$$6 = 2^b - 2^a = 2^a\left(2^{b-a} - 1\right).$$

2^a and $2^{b-a} - 1$ are both integers so are both factors of 6. If $a \geqslant 2$ then the right hand side is divisible by 4 while 6 is not. So $a = 1$ or $a = 0$. If $a = 0$ then $6 = 2^b - 1$ which is impossible. Finally if $a = 1$ then $3 = 2^{b-1} - 1$ so $b = 3$: this gives $n = 5$ and so $m = 4$. Note that $2^4 + 9 = 25 = 5^2$ so we have indeed found a solution.

Hence the only solution is $m = 4$ and $n = 5$.

b) We wish to solve $2^m + 5 = n^2$. In the last part we started by doing some algebra (difference of two squares) to make progress. Here that does not seem to work so well so we might start by seeing if any number-theoretic ideas will be helpful.

We would like to use modular arithmetic but it is not immediately obvious which modulus we should use. It is often a good idea to consider an equation modulo a number so as to remove a term of the equation. So we could consider the equation modulo 5: $2^m \equiv n^2 \pmod{5}$. However, 2^m can be anything modulo 5 so this is not so helpful. Another option is to consider the equation modulo some power of 2 so that the 2^m term disappears if m is large enough. Let us start by considering small m:

- if $m = 0$ then $n^2 = 6$ which is not possible;
- if $m = 1$ then $n^2 = 7$ which is not possible;
- if $m = 2$ then $n^2 = 9$ so $n = 3$ (note we are told $n \geqslant 0$).

Otherwise m is at least 3 so 8 divides 2^m. Considering the equation modulo 8, we get

$$n^2 \equiv 5 \pmod{8}.$$

Now we can calculate that the squares $0^2, 1^2, \ldots, 7^2$ are 0, 1 or 4 $\pmod{8}$. So $n^2 \equiv 5 \pmod{8}$ has no solutions.

Thus the only solution is $m = 2$ and $n = 3$.

4. In previous sheets we have seen inequalities which could have been proved using the AM-GM inequality. The inequality says that for *non-negative* real numbers $a_1, a_2, \ldots, a_n$, their arithmetic mean (AM) is always at least their geometric mean (GM):

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \cdots a_n},$$

and furthermore equality occurs if and only if $a_1 = a_2 = \cdots = a_n$. Let us prove this by varying the a_i while keeping their arithmetic mean fixed.

a) Suppose that the arithmetic mean of the a_i is m. Show that if the a_i are not all equal then one of them must be strictly less than m and another must be strictly greater than m.

b) Suppose that $a_s < m < a_t$. Show that if we replace a_s by m and a_t by $a_s + a_t - m$ then the arithmetic mean of the a_i has not changed but the geometric mean has strictly increased.

c) Deduce the AM-GM inequality. Have you shown that equality occurs if and only if all the a_i are equal?

d) x, y and z are non-negative real numbers which sum to 1. What is the largest possible value of xyz?

Solution

Repeatedly altering variables is a very natural and useful technique for solving some inequalities. Here we are repeatedly making the variables equal to m which is the case of equality.

First we get a small detail out of the way: if some of the a_i are 0, then it is easy to verify the AM-GM inequality directly since the geometric mean is 0. For the rest of the solution we will assume that the a_i are all positive.

a) We are told that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = m.$$

It follows that their sum, $a_1 + a_2 + \cdots + a_n$, is mn. If the a_i are all equal then they must all be equal to m. Otherwise at least one of the a_i is not equal to m.

Suppose that at least one of the a_i is strictly greater than m. If the others are all at least m then their sum is strictly greater than mn which is absurd. Hence one of the a_i must be strictly less than m.

Conversely suppose that at least one of the a_i is strictly less than m. If the others are all at most m then their sum is strictly less than mn which is absurd and so one of the a_i must be strictly greater than m.

Putting this all together: either the a_i are all equal (and so all equal to m) or at least one is greater than m and at least one is less than m.

b) Suppose that $a_s < m$ and $a_t > m$. We replace a_s by m and a_t by $a_s + a_t - m$. Now $m + (a_s + a_t - m) = a_s + a_t$ so the sum of the a_i has not changed and hence their arithmetic mean has not changed.

Let us see what has happened to the geometric mean of the a_i. All that has changed in the expression for the geometric mean is that $a_s a_t$ has become $m(a_s + a_t - m)$. Since all the a_i are *positive*, in order to prove the geometric mean has strictly increased it is enough to show that

$$a_s a_t < m(a_s + a_t - m),$$

which is equivalent to

$$0 < ma_s + ma_t - a_s a_t - m^2.$$

We can factorise the right-hand side as $(m - a_s)(a_t - m)$. $m > a_s$ and $a_t > m$ so $ma_s + ma_t - a_s a_t - m^2$ is indeed positive.

c) Suppose we have n positive real numbers $a_1, a_2, \ldots, a_n$. Let their arithmetic mean be m. If the a_i are all equal then they are all equal to m and their arithmetic and geometric means are both m so we have equality in AM-GM.

Otherwise, by part *a*) we can find $a_s < m$ and $a_t > m$. Then we replace a_s by m and a_t by $a_s + a_t - m$. By part *b*), this keeps the arithmetic mean of the a_i unchanged (equal to m) and strictly increases the geometric mean of the a_i. We repeatedly apply this procedure: after each application at least one more of the a_i is equal to m. So, certainly, after at most n (in fact $n - 1$ is enough) steps all the a_i will be equal to m.

We have equality at the end and throughout we strictly increased the geometric mean of the a_i while keeping their arithmetic mean fixed. Hence the AM-GM inequality holds and we have equality only if all the a_i are equal.

d) We are told $x + y + z = 1$ and wish to find the largest possible value of xyz. By the AM-GM inequality for three variables,

$$\sqrt[3]{xyz} \leqslant \frac{x+y+z}{3} = \frac{1}{3},$$

so $xyz \leqslant \left(\frac{1}{3}\right)^3 = \frac{1}{27}$. We must check that we can actually attain $\frac{1}{27}$ to conclude that this is the largest possible value of xyz. To get equality in AM-GM we need $x = y = z$ and indeed if $x = y = z = \frac{1}{3}$ then $x + y + z = 1$ and $xyz = \frac{1}{27}$. Therefore the largest possible value of xyz is $\frac{1}{27}$.

5. Find all positive integers x and y with $x \geqslant y$ such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{597}.$$

Solution

The equation will look more manageable if we multiply up by $597xy$ to obtain $597y + 597x = xy$ or equivalently $xy - 597x - 597y = 0$. This contains one of those expressions that looks almost, but not quite, factorisable. By a trick used previously (see, for example, *Question 1 of Sheet 3*) we can obtain the equation in the form

$$xy - 597x - 597y + 597^2 = (x - 597)(y - 597) = 597^2.$$

This implies that each bracket must be a factor of 597^2. Now $597 = 3 \times 199$, so $597^2 = 3^2 \times 199^2$. Since 199 as well as 3 is prime, it follows that 597^2 has $(2 + 1)(2 + 1) = 9$ factors. These are

$$1 \quad 3 \quad 199 \quad 3^2 \quad 199^2 \quad 3\times 199 \quad 3^2\times 199 \quad 3\times 199^2 \quad 3^2\times 199^2.$$

As a preliminary it will be useful to calculate $199^2 = (200 - 1)^2 = 40000 - 400 + 1 = 39601$. Pairing the factors we have the following results.

a) $x - 597 = 3^2 \times 199^2$ and $y - 597 = 1$. Then $x = 9 \times 39601 + 597 = 356409 + 597$ and $y = 598$. Completing the calculation, $x = 357006$ and $y = 598$.

b) $x - 597 = 3 \times 199^2$ and $y - 597 = 3$. Then $x = 3 \times 39601 + 597 = 118803 + 597$ and $y = 600$. Completing the calculation, $x = 119400$ and $y = 600$.

c) $x - 597 = 3^2 \times 199$ and $y - 597 = 199$. Then $x = 1791 + 597$ and $y = 796$. Completing the calculation, $x = 2388$ and $y = 796$.

d) $x - 597 = 199^2$ and $y - 597 = 3^2$. Then $x = 39601 + 597 = 40198$ and $y = 606$.

e) $x - 597 = y - 597 = 3 \times 199 = 597$. Then $x = y = 597 + 597 = 1194$.

Strictly speaking, the answers found are candidates for solutions and ideally we should check that each pair does in fact provide a solution. One example is given: let us check that the pair $x = 119400$ and $y = 600$ does in fact satisfy the equation. Substituting these values gives

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{119400} + \frac{1}{600} = \frac{1}{2\times 597\times 100} + \frac{1}{6\times 100} = \frac{3+597}{6\times 597\times 100} = \frac{600}{597\times 600} = \frac{1}{597},$$

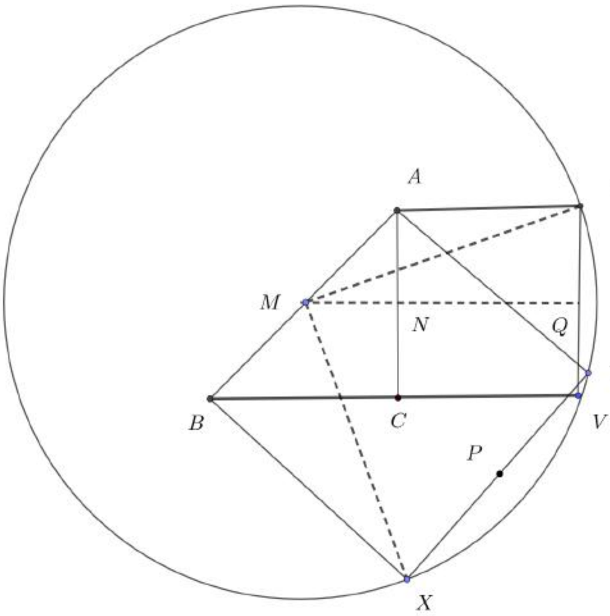
as required. Notice that this check involves as much factorisation, and as little multiplying out, as possible.

Some more history: what is special about the integer 597? What is generally agreed to be the oldest secondary school in England is the successor, in an unbroken sequence, to the monastic educational establishment founded by St Augustine in Canterbury in 597 CE.

6. Consider two squares $ABXY$ and $ACVW$, positioned so that $\angle ACB = 90°$. The length AB is 12. The points X, Y, V and W all lie on a circle; furthermore all the points mentioned in the question lie either inside or on this circle. What is the perimeter of triangle ABC?

SOLUTION

As ever, we start by drawing a clear diagram. This may require several attempts as the conditions of the question mean that this diagram is tightly constrained. The figure below illustrates the configuration. You may notice that the diagram does not look quite right but this will be explained later in the solution.



If you imagine the diagram without the dotted lines, there is not much to go on. There is a right-angled triangle so Pythagoras' theorem can be involved in finding the lengths of its other sides. Using the standard notation for the sides of the triangle ABC, denote the length of AC by b and that of BC by a. Then by Pythagoras' theorem,

$$a^2 + b^2 = 144. \tag{1}$$

Let M be the midpoint of AB and N the midpoint of AC. Since $\angle ABC$ is a right angle, the circle with diameter AB and centre M (not shown) passes through C. The line segment AC is a chord of this circle so its perpendicular bisector passes through M. This perpendicular bisector is MN. If MN is extended to meet VW at Q then, since $ACVW$ is a square, MQ is the perpendicular bisector of the chord VW of the circle, S say, through X, Y, V and W. Let P be the midpoint of XY. Since $ABXY$ is a square, MP is the perpendicular bisector of AB and XY. However, XY is a chord of S so we see that the perpendicular bisectors of the chords VW and XY both pass through M. It follows that M is the centre of the circle S.

We can now proceed to further calculation and find another equation linking a and b. Because M is the midpoint of AB, $BM = 6$. Pythagoras' theorem applied to triangle BXM gives $XM^2 = 6^2 + 12^2 = 6^2(1+4)$. It follows that $XM = 6\sqrt{5}$. This tells us that the radius of the circle is $6\sqrt{5}$ and so $WM = 6\sqrt{5}$.

Now apply Pythagoras' theorem to the right- angled triangle WMQ. First note that $QM = \frac{1}{2}a + b$ and so

$$\left(\tfrac{1}{2}a + b\right)^2 + \left(\tfrac{1}{2}b\right)^2 = 36 \times 5. \tag{2}$$

On rearranging, this becomes

$$\tfrac{1}{4}\left(a^2 + b^2\right) + ab + b^2 = 36 \times 5. \tag{3}$$

Substituting from (1), this becomes

$$36 + ab + b^2 = 36 \times 5. \tag{4}$$

We now have $a^2 + b^2 = 144$ from (1) and $ab + b^2 = 144$ from (4). Equating these yields $a^2 + b^2 = ab + b^2$, implying $a^2 = ab$. Since $a > 0$, it may be cancelled to show that $a = b$.

The equation (1) may now be written as $2a^2 = 144$, giving $a = 6\sqrt{2} = b$. We conclude that the perimeter of triangle ABC is $12 + 6\sqrt{2} + 6\sqrt{2} = 12(1 + \sqrt{2})$.

Remark: The reason for the odd aspect of the diagram is that the points V and Y actually coincide. You may like to have a go at producing an argument to prove this.

7. Place the following numbers in increasing order of size and justify your reasoning:

$$3^{3^4}, \quad 3^{4^3}, \quad 3^{4^4}, \quad 4^{3^3}, \quad 4^{3^4}.$$

Note that $a^{b^c} = a^{(b^c)}$.

[British Mathematical Olympiad Round 1 2014 Question 1]

SOLUTION

The five numbers are 3^{81}, 3^{64}, 3^{256}, 4^{27} and 4^{81}. Clearly

$$3^{64} < 3^{81} < 3^{256}.$$

Now $4^{27} = (4^3)^9 = 64^9$. Looking for a power of 3 fairly near 64, it is sensible to select $3^4 = 81$ and to note that $3^{64} = (3^4)^{16} = 81^{16}$. Now $64 < 81$ and $9 < 16$. Hence $64^9 < 81^{16}$. We now have

$$4^{27} < 3^{64} < 3^{81} < 3^{256}.$$

All that remains is to fit 4^{81} into the sequence. Clearly this number is greater than 3^{81}, so now we have to compare 4^{81} and 3^{256}. However, the former number is 64^{27} and the latter is 81^{64}. By a similar argument to that used previously, $64^{27} < 81^{64}$. Hence the increasing sequence is

$$4^{27} < 3^{64} < 3^{81} < 4^{81} < 3^{256}.$$

Thus, in increasing order of size and in their original form, the five numbers are:

$$4^{3^3}, \quad 3^{4^3}, \quad 3^{3^4}, \quad 4^{3^4}, \quad 3^{4^4}.$$

8. *A graph consists of some points (the vertices) and line segments (the edges) joining some pairs of points. If there are n vertices and an edge joins each pair of points, then we have* ***a complete graph with n vertices.***

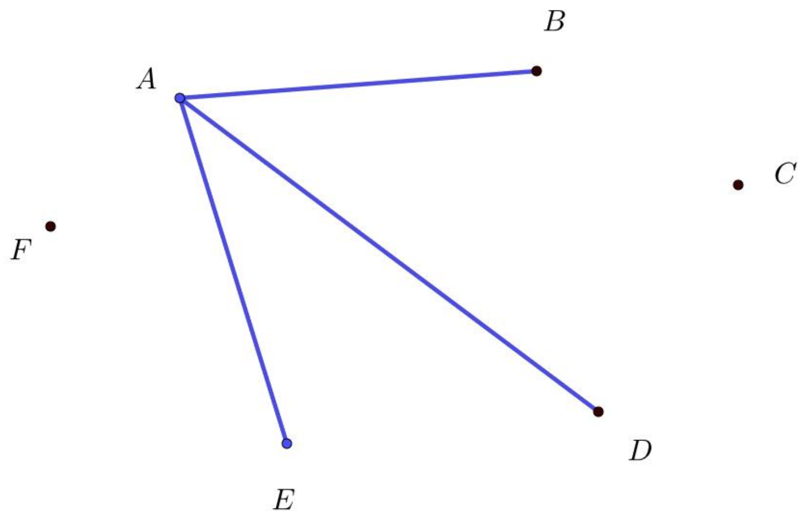
a) Consider the vertices of a hexagon and join each pair of points by a line segment. The edges of this graph are coloured either red or blue. Show that the graph contains a triangle whose sides are all of the same colour.

b) In the holidays each student from a very enthusiastic group of 17 mathematicians emails all the others about algebra, geometry or statistics. Each pair corresponds about exactly one topic but any student may correspond with different colleagues about different topics. For instance, Clare and Paul may correspond about algebra while Clare and Charlotte write to each other about statistics. Prove that there are at least 3 students who all correspond with each other on the same topic.

Solution

a) To give yourself a feel for the complete graph described, draw a diagram with a particular colouring for yourself. Note that the vertices need not be the vertices of a regular hexagon: this question is not about lengths and angles but about counting and relationships. Though it is not relevant to the problem, you might want to consider how many edges the graph has.

On first inspection, there is no obvious way to break into the argument. To make a start, focus on one particular vertex and consider the edges leaving it. There are five of these and there must be at least three of the same colour. This is a crucial step, so make sure that you understand why this statement is true. (What would be the consequence of having at most two red edges and two blue edges leaving the vertex?) Without loss of generality*, we may assume that these three edges are blue. The situation is illustrated in the diagram below: the vertex considered is A and three blue edges are shown. Note that the colour of AC and that of AF are irrelevant; what matters is that we have at least three blue edges leaving A.



If at least one of the edges BD, BE or DE is blue, then a blue triangle is formed. On the other hand, if this does not happen, then these three edges must all be red, forming the red triangle

BDE. The result is proved.
**The phrase "without loss of generality"was explained in the solution to Question 3 of Hanna Neumann Sheet 2.*

b) This part of the question appears to have nothing to do with the previous result. However, a complete graph provides a powerful representation of the problem. Moreover, the previous argument can be adapted to answer this new question. First, how does the representation work? The 17 students are the vertices of the graph. Since each pair of students is in correspondence, each pair of vertices is joined by an edge. Thus we have a complete graph with 17 vertices. There are 3 topics involved in the correspondence, so each edge is coloured in one of 3 colours, say red, blue and green.

Start by focussing on one vertex S, say. There are 16 edges leaving it and among these must be at least 6 edges of the same colour. (Again check that you understand why this statement is true.) Without loss of generality, let this colour be blue. Pick 6 of these blue edges and suppose that they connect S to the vertices A_1, A_2, A_3, A_4, A_5 and A_6. Since these vertices are part of the complete graph, they are all connected to each other and each of them is connected to S. If any edge A_iA_j, with $i \neq j$, is blue then SA_iA_j is a blue triangle and we are done. If this does not happen, then all the edges A_iA_j are coloured either red or green. Since they are all connected to each other they form a complete graph with 6 vertices that is coloured with 2 colours. But this is the situation in part *a)* so the graph with vertices A_1, A_2, A_3, A_4, A_5, A_6 contains a triangle whose sides are all the same colour. Returning to the original formulation, this triangle represents 3 students who are corresponding on the same topic.
Remark: In this question the phrases at least 3 edges of the same colour *and then* at least 6 edges of the same colour *can be considered as special cases of the pigeonhole principle. This was stated at the end of the solution to Question 5 on Sheet 2.*

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Hanna Neumann

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after Hanna Neumann (1914–1971).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Neumann_Hanna.html for more information, or search "Hanna Neumann mathematician" to find a wider range of sources.

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1. A long time ago Evelyn went into a very old-fashioned pet shop where the window display consisted of a basket of kittens and a mini-aviary of budgies. The shop owner said that the number of kittens in the basket multiplied by the number of budgies was equal to the total number of legs in the window. Also the sum of the number of kittens and the number of budgies was odd. Since every animal had the usual number of legs, Evelyn was able to work out the total number of animals in the window. What was this total?

Solution

Since the total number of animals is odd, there must be an even number of kittens and an odd number of budgies, or vice versa. Considering the first case, let the number of kittens be $2k$ and the number of budgies be $2b + 1$. Then

$$2k(2b + 1) = 4 \times 2k + 2(2b + 1). \tag{1}$$

On cancelling, this yields $k(2b + 1) = 4k + 2b + 1$ or

$$(2b + 1)(k - 1) = 4k. \tag{2}$$

(*Algebra note:* when factorising, it is usually a good idea to do so as comprehensively as possible.) In (2) the right-hand side is divisible by 4 and so must the left-hand side be. Now $2b + 1$ is odd and cannot possibly be divisible by 4; it follows that 4 divides $k - 1$. Set $k - 1 = 4m$, which in turn implies that $k = 4m + 1$. Now (2) becomes $(2b + 1) \times 4m = 4k$. On cancelling and replacing k by $4m + 1$, we have

$$m(2b + 1) = 4m + 1. \tag{3}$$

Recall that m and b will, eventually, give us the numbers of animals. Thus it makes sense to get the terms in these unknowns on the left-hand side of the equation and doing so yields

$$m(2b - 3) = 1. \tag{4}$$

This shows that $m = 2b - 3 = 1$ or $m = 1$ and $b = 2$. Thus the number of budgies is $2b + 1 = 5$ and the number of kittens is $2k = 2(4m + 1) = 2(4 \times 1 + 1) = 10$. There are 15 animals altogether.

Is the solution complete? Well no, because the animals do not have the same number of legs and the case where the number of kittens is odd and the number of budgies is even must be considered. However, a similar argument can be constructed. With $2b$ budgies and $2k + 1$ kittens, the initial equation is

$$2b(2k + 1) = 2 \times 2b + 4(2k + 1) = 4(b + 2k + 1). \tag{5}$$

Considering b on the left-hand side and the factor 4 on the right-hand side means that b must be even. (Why?) Setting $b = 2m$ in (5) and cancelling yields $m(2k + 1) = 2m + 2k + 1$. Factorising as before: $(2k + 1)(m - 1) = 2m$. It follows that $m - 1$ must be even. (Why?) Setting $m - 1 = 2n$, then cancelling, produces $n(2k + 1) = (2n + 1)$. This rearranges to $n(2k - 1) = 1$ with solution $n = k = 1$. This gives a different solution where the number of budgies is 12 and the number of kittens is 3. However the number of animals is still 15.

Alternative solution: this is given (in outline) as it has a completely different sort of shape and the two solutions emerge in a different way. Let the number of budgies be B and the number of kittens be K. The shop owner tells Evelyn that

$$BK = 2B + 4K \tag{6}$$

and $B + K$ is odd.

To make use of (6) we use the factorisation trick that we have seen before (e.g. in Question 1 of Sheet 3 and Question 5 of Sheet 4),

$$(B-4)(K-2) = BK - 2B - 4K + 8 = 8,$$

so $B-4$ and $K-2$ are two integers multiplying to 8. The possibilities are

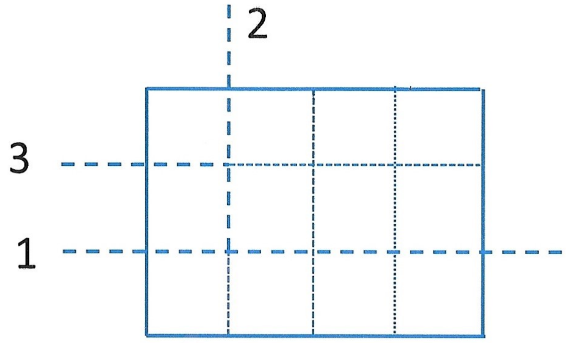
$$8 = -1 \times -8 = -2 \times -4 = 2 \times 4 = 1 \times 8.$$

Now it remains to check all the possibilities against the conditions that B and K are positive and that $B + K$ is odd. Happily these checks leave the two previous solutions: 5 budgies and 10 kittens or 12 budgies and 3 kittens. In both cases the number of animals is 15.

2. A rectangular bar of chocolate is made up of mn 1×1 squares of chocolate with m squares in one direction and n in the other. The bar is to be divided into its constituent squares. At each step you may pick up just one of the pieces so far produced and break it once along any of its vertical or horizontal divisions. What is the number of cuts needed to divide the bar into its mn squares?

SOLUTION

The diagram shows a possible start for the 4×3 case.



The first break is along the horizontal division labelled 1, giving two pieces. Then imagine that you pick up the upper piece and break it along the division labelled 2; there are now three pieces. Pick up the left-hand part of the upper section and break it along the division labelled 3, producing four pieces altogether.

The pattern is: after 0 breaks there is 1 piece of chocolate; after 1 break there are 2 pieces; after 2 breaks there are 3 pieces; after 3 breaks there are 4 pieces and so on. The process continues with the number of pieces being 1 more than the number of breaks. Thus it takes $mn - 1$ breaks to reduce the bar to its mn constituent squares. We can express what happens at each stage by:

$$\text{number of pieces of chocolate } - \text{ number of breaks} = 1.$$

This is an example of an *invariant*, defined in the solution to Question 3 of Sheet 3.

Remark 1: In previous questions we have seen cases where "and so on" arguments can be formalised by using induction. However, it is hardly needed in this simple case.
Remark 2: Mathematical folklore recounts instances of eminent mathematicians who were apparently unable to solve this "simple" problem.

3. *More about modular arithmetic: you may want to look back at Questions 2 and 3 of Sheet 4, where the idea was introduced.*

a) Let a and n be positive integers and suppose that r is the remainder when a is divided by n. Show that $a \equiv r \pmod{n}$.

b) Let a, b and c be positive integers and let n be a positive integer such that c and n have no common factor other than 1*. Show that, if $ca \equiv cb \pmod{n}$, then $a \equiv b \pmod{n}$. **In this situation we usually say that c and n are* ***coprime*** *or* ***relatively prime***.

c) Now p is a prime and a is a positive integer such that p and a are coprime. Explain why the integers $a, 2a, 3a, \ldots, (p-1)a$ are congruent modulo p to $1, 2, 3, \ldots, p-1$ in some order.

d) Again p is a prime and a is a positive integer such that p and a are coprime. Prove Fermat's Little Theorem, which states that $a^{p-1} \equiv 1 \pmod{p}$.

e) Use Fermat's Little Theorem to find the units digit of 3^{100}.

Solution

a) The integer a can be expressed as $a = kn + r$. This is equivalent to $a - r = kn$. Hence $a \equiv r \pmod{n}$.

b) Since $ca \equiv cb \pmod{n}$, we know that $ca - cb = c(a - b)$ is divisible by n. Because of the coprime condition, n cannot divide c. Therefore n divides $a - b$ and the result follows.

c) First note that none of the integers $a, 2a, 3a, \ldots, (p-1)a$ can be congruent to 0 modulo p. If it were true that $ra \equiv 0 \pmod{p}$ for some r satisfying $1 \leqslant r \leqslant p-1$, then it would follow that p divides ra. This is impossible because of the coprime condition. Secondly, note that each ra must be congruent modulo p to one of $1, 2, 3, \ldots, p-1$. This statement follows from part *a)* since $0, 1, 2, 3, \ldots, p-1$ are the possible remainders on division by p and the case of 0 has already been considered. The desired result will follow if we can be sure that all the ra are congruent to different remainders. Suppose we have $ra \equiv sa \pmod{p}$, where $1 \leqslant r < s \leqslant p-1$. Then by part *b)* we could cancel a to obtain $r \equiv s \pmod{p}$, which is impossible. The desired result follows.

d) From part *c)* we have $p-1$ congruences linking the integers $a, 2a, 3a, \ldots, (p-1)a$ and the integers $1, 2, 3, \ldots, (p-1)$. Multiplying them together, which is valid for congruences, yields

$$1 \times 2 \times 3 \times \ldots \times (p-1)a^{p-1} \equiv 1 \times 2 \times 3 \times \ldots \times (p-1) \pmod{p}.$$

Because p and $1 \times 2 \times 3 \times \ldots \times (p-1)$ are coprime, part *b)* may be used to cancel the latter. This gives $a^{p-1} \equiv 1 \pmod{p}$.

e) The first thing to do is to decide what the value of p should be. It should be coprime to 3 and it needs to be related to 100. The prime factors of 100 are 2 and 5 but since $2 - 1 = 1$, clearly 5 is the value to try. Fermat's Little Theorem then gives

$$3^{100} = (3^{25})^4 = (3^{25})^{5-1} \equiv 1 \pmod{p}.$$

Now recollect that the units digit of an integer is the remainder on division by 10, so it would be productive to convert the congruence into a statement about remainders. This corresponding statement is that $3^{100} = 5k + 1$, where k is a suitable positive integer. Finally we need to get 10 into the picture and so it would be convenient if k were even. However, if k were odd, then $5k + 1$ would be even. Since 3^{100} is odd, this furnishes a ***contradiction***. Therefore k is indeed even and we can set $k = 2m$. This yields $3^{100} = 10m + 1$, which implies that the units digit of 3^{100} is 1.

Remark: The last part of the question gave an opportunity to apply Fermat's Little Theorem. However, in the example considered, there is an alternative very simple solution. Hint: note that $3^{100} = (3^4)^{25}$ *and* $3^4 = 81$.

4. *a*) Is it possible to find three points in the plane such that the distance between each pair of them is 1? Is it possible to find four points in the plane such that the distance between each pair of them is 1?

b) Is it possible to find four points in (three-dimensional) space such that the distance between each pair of them is 1? What about five points?

Solution

a) We can certainly find three such points: take the three points to be the vertices of an equilateral triangle of side length 1. Indeed, if we have three points that satisfy the condition, then they *must* form an equilateral triangle (from the definition). This is important because it means that, in trying to prove an arrangement with four points is impossible, we can assume that the first three form an equilateral triangle. Thus one way to finish off part *a)* is to note that the only point which is equidistant from all three vertices of a triangle is its circumcentre. However the distance from the circumcentre to the vertices of the equilateral triangle with side length 1 is only __what?

b) In three dimensions, we can take the four points to be the vertices of a regular tetrahedron. For five points, first consider four points A, B, C and D such that the distance between each pair is 1. Then we can find a plane Σ containing A, B and C; from part *a)* these points must form an equilateral triangle in Σ. Now let P be the closest point of Σ to D. (This is called the *projection* of P onto Σ and is the foot of the perpendicular from D onto that plane.) Pythagoras's theorem tells us that $AD^2 = AP^2 + PD^2$ and $BD^2 = BP^2 + PD^2$ and $CD^2 = CP^2 + PD^2$. However, $AD = BD = CD = 1$, so $AP = BP = CP$. Hence P is the circumcentre of the triangle ABC, giving $AP =$ __ and $DP =$ __ (fill in the blanks for yourself). Hence there are only two possible positions for D, (depending on which side of Σ it is). These positions are __ apart and so finding a fifth point is impossible.

5. Prove that for all positive real numbers x, y and z

a) $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geqslant 3$;

b) $(x + y)(y + z)(z + x) \geqslant 8xyz$;

c) $\frac{1}{x^2} \geqslant 3 - 2x$;

and determine any cases where equality holds.

d) Prove that for any positive integer $n \geqslant 2$: $\left(\frac{n+1}{2}\right)^n > n!$.

Solution

There are many ways to prove these inequalities but it is useful to recall the AM-GM inequality from Question 4 of Sheet 4: for any non-negative real numbers $a_1, a_2, \ldots, a_n$

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

AM-GM essentially allows us to compare sums (AM) with products (GM). Thus if we are asked to prove an inequality of the form "sum $\geqslant$ product" then it is worth considering AM-GM.

a) The product of $\frac{x}{y}$, $\frac{y}{z}$ and $\frac{z}{x}$ is 1 so it very much looks like AM-GM is a good way to go. x, y and z are positive so $\frac{x}{y}$, $\frac{y}{z}$ and $\frac{z}{x}$ are positive. By AM-GM

$$\frac{1}{3}\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right) \geqslant \sqrt[3]{\frac{x}{y} \times \frac{y}{z} \times \frac{z}{x}} = 1.$$

Multiplying by 3 gives us what we want. Now for equality in our application of AM-GM we need $\frac{x}{y} = \frac{y}{z} = \frac{z}{x}$. Furthermore equality means $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 3$ so in fact we would need $\frac{x}{y} = \frac{y}{z} = \frac{z}{x} = 1$. This gives $x = y = z$ and we do indeed get equality in this case.

b) The left-hand side of this inequality is a product of sums and the right-hand side is a product. We could multiply out the left-hand side simplify everything and then try and apply AM-GM (and this does indeed work). However, here we will take a slightly different approach: we will apply AM-GM to each of the three sums on the left-hand side. We have

$$\frac{x+y}{2} \geqslant \sqrt{xy},$$
$$\frac{y+z}{2} \geqslant \sqrt{yz},$$
$$\frac{z+x}{2} \geqslant \sqrt{zx}.$$

Multiplying the three inequalities together (note both sides are positive in all of them) gives

$$\tfrac{1}{8}(x+y)(y+z)(z+x) \geqslant xyz.$$

Equality occurs if and only if we have equality in all three of our AM-GM inequalities which occurs if and only if $x = y$, $y = z$ and $z = x$. Thus the equality case is $x = y = z$.

c) We want to prove $\frac{1}{x^2} + 2x \geqslant 3$. The left-hand side is a sum so we might try and apply AM-GM. If we apply it to $\frac{1}{x^2}$ and $2x$ then we get

$$\frac{1}{2}\left(\frac{1}{x^2} + 2x\right) \geqslant \sqrt{\frac{2}{x}},$$

which is not so helpful. The problem is that multiplying $\frac{1}{x^2}$ and $2x$ does not cancel out all the x's (compare with part *a)*).

The way around this is to write $2x$ as $x + x$ which is particularly helpful as $\frac{1}{x^2} \times x \times x = 1$. Applying AM-GM to $\frac{1}{x^2}$, x and x gives

$$\frac{1}{3}\left(\frac{1}{x^2} + 2x\right) \geqslant \sqrt[3]{\frac{1}{x^2} \times x \times x} = 1.$$

For equality we need $\frac{1}{x^2} = x$ i.e. $x^3 = 1$ so $x = 1$ and indeed we do get equality in this case.

d) $n! = 1 \times 2 \times 3 \times \cdots \times n$ is a product so AM-GM looks to be a good thing to try. Applying AM-GM to $1, 2, \ldots, n$ gives

$$\sqrt[n]{n!} \leqslant \frac{1 + 2 + \cdots + n}{n} = \frac{n+1}{2},$$

where we have used the fact that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Raising both sides to the power of n gives $n! \leqslant \left(\frac{n+1}{2}\right)^n$ which is almost what we want: we need to just show that equality cannot occur.

Equality would occur if and only if $1 = 2 = \cdots = n$ but as $n \geqslant 2$ this is not the case and so we do have a strict inequality.

6. The squares of a $1 \times N$ games board are numbered $1, 2, \ldots, N$ from the left. Six counters, that are labelled A, B, C, D, E, F, are placed on the six leftmost squares. If any counter has an empty adjacent square to its right, then it can be moved to that square. If a counter has an adjacent counter to its right, then it is allowed to jump over that adjacent counter, as long as the next square is empty. Moves to the left are not permitted. Find the minimum value of N for which the counters end up further along the board, in six successive squares but in the reverse order.

SOLUTION

To achieve the desired outcome, the board has to be, in some sense big enough. What might a board that is too small look like? Note that if $N \leqslant 10$, then because of the ban on leftwards movement F has to remain where it is. There are then at most four squares to the right of F, into which the five remaining counters cannot be fitted.

Next consider the case $N = 11$, which is illustrated below. Again F has to stay where it is, so E has to jump over it.

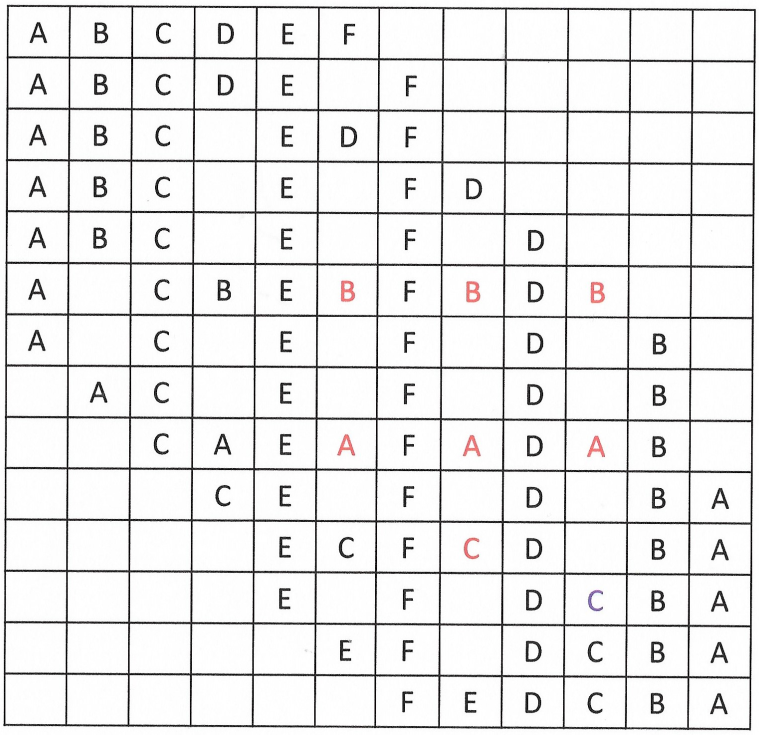
A	B	C	D	E	F					
A	B	C		D	F	E				
A	B	C	D		F	E				
A	B		D	C	F	E				

This is shown in the second row of the diagram; additionally D has moved into the vacated space to its right. However, now no further move is possible. In the third row of the diagram D has been returned to its previous position. Now the only other possible move is for C to jump over D. The fourth row now shows that an impasse of the previous sort occurs and we conclude that the case $N = 11$ does not work.

The case $N = 12$ might be more productive because F does not have to stay in its original position. The board is illustrated in the next diagram.

A	B	C	D	E	F						
A	B	C	D		E	F					
A	B	C		D		F	E				
A	B	C			D	F	E				

In the second row of the diagram F and E have both moved to the right. Then in the third row E has jumped over F and D has moved to its right. The only possible move now is for D to move to the right again, only to enter into another jam of three counters together. This suggests that, after F, we should move a counter other than E; let us try D. This is illustrated by the third diagram shown below.



A	B	C	D	E	F						
A	B	C	D	E		F					
A	B	C		E	D	F					
A	B	C		E		F	D				
A	B	C		E		F		D			
A		C	B	E	B	F	B	D	B		
A		C		E		F		D		B	
	A	C		E		F		D		B	
		C	A	E	A	F	A	D	A	B	
			C	E		F		D		B	A
				E	C	F	C	D		B	A
				E		F		D	C	B	A
					E	F		D	C	B	A
						F	E	D	C	B	A

In the second row of the diagram F has moved to its final position; in the third and fourth rows D jumps over E and then over F. To gain its final position it must now move one square to the right and this is shown in the fifth row. We have dealt with D after F, so can we now move B after D? In the next row B has jumped over C and the the red B's show its subsequent positions as it jumps over E, F and D. Then in the next (the seventh) row, B moves one square right to its final position. The process is completed in the remaining rows of the table: first A is moved, then C and then E. As before red letters show the positions after intermediate jumps. Note that E has to move one place to the right (see the penultimate row) before it can jump into its final position.

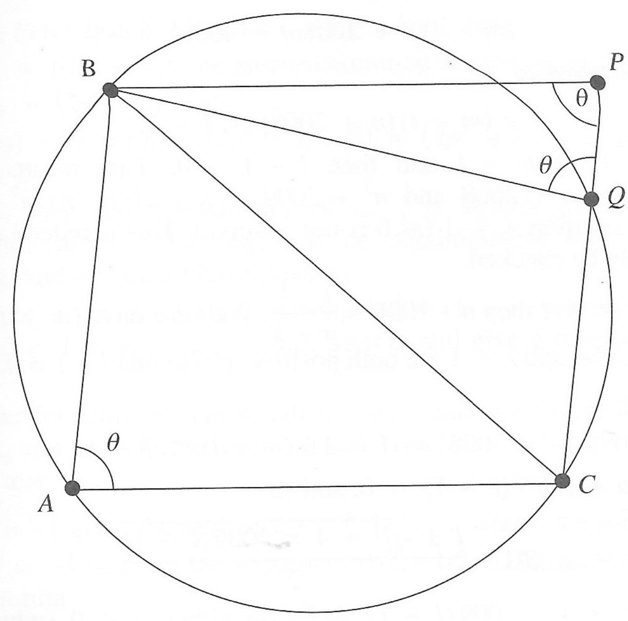
Thus the minimum value of N for which the desired outcome can be achieved is $N = 12$.

7. Let $ABPC$ be a parallelogram such that ABC is an acute-angled triangle. The circumcircle of triangle ABC (that is, the unique circle through its vertices) meets the line CP again at Q. Prove that $PQ = AC$ if and only if $\angle BAC = 60°$.

[British Mathematical Olympiad Round 1 2008 Question 3]

SOLUTION

The essential diagram appears below. A numerical answer is involved, so it will be useful to set $\measuredangle BAC = \theta$. Since $ACQB$ is a cyclic quadrilateral, $\measuredangle CQB = 180° - \theta$. It follows immediately that $\measuredangle PQB = \theta$ by angles on a straight line. It is often useful to make the jump from $\measuredangle BAC$ to $\measuredangle PQB$ directly and this result is expressed by saying that the *external angle of a cyclic quadrilateral is equal to the opposite internal angle.*



So we have $\measuredangle PQB = \theta$ since the external angle of the cyclic quadrilateral $ACQB$ is equal to the opposite internal angle. Further, opposite angles of a parallelogram are equal so $\measuredangle QPB = \theta$. Hence triangle BQP is isosceles and therefore $BP = BQ$. It is also the case that opposite sides of a parallelogram are equal and so $BP = AC$.

As a consequence of these results, $AC = PQ$ implies that $PQ = AC = BP = BQ$, making triangle BQP equilateral. Hence in this triangle the two angles marked θ are equal to 60°. It follows that $\measuredangle BAC = \theta = 60°$. The implication goes the other way: if $\theta = \measuredangle BAC = 60°$, then triangle BQP is equilateral. Hence $AC = BP = PQ$. We can say of these opposite implications that $AC = PQ$ if and only if $\measuredangle BAC = 60°$ and this is the required result.

8. x, y and z are real numbers satisfying the following simultaneous equations:

$$x^3 - xyz = 2, \quad y^3 - xyz = 6, \quad z^3 - xyz = 20.$$

What possible values can $x^3 + y^3 + z^3$ take?

SOLUTION

The desired expression contains cubed terms only but xyz appears in each equation. If we wrote the cubes in terms of xyz, then we would be able to form the cube of xyz. Rearranging the equations,

$$x^3 = xyz + 2, \quad y^3 = xyz + 6, \quad z^3 = xyz + 20,$$

and multiplying together we get

$$x^3y^3z^3 = (xyz + 2)(xyz + 6)(xyz + 20).$$

To simplify the notation we set $xyz = K$ and obtain the cubic equation

$$K^3 = (K + 2)(K + 6)(K + 20).$$

This is equivalent to $K^3 = K^3 + 28K^2 + 172K + 240$ or $28K^2 + 172K + 240 = 0$. Dividing by 4 yields $7K^2 + 43K + 60 = 0$ and this factorises as $(7K + 15)(K + 4) = 0$, giving the roots $K = -4$ and $K = -\frac{15}{7}$.

The case $K = -4$: substituting in the original equations gives $x^3 = -2$, $y^3 = 2$ and $z^3 = 16$. The sum of these values is 16.

The case $K = -\frac{15}{7}$: substituting in the original equations gives $x^3 = -\frac{1}{7}$, $y^3 = \frac{27}{7}$ and $z^3 = \frac{125}{7}$. The sum of these values is $\frac{151}{7}$.

We should check that both of these solutions are valid. For the first one we have $x = \sqrt[3]{-2} = -\sqrt[3]{2}$, $y = \sqrt[3]{2}$ and $z = \sqrt[3]{16} = 2\sqrt[3]{2}$ (which are all real) and so $xyz = -4$. These do satisfy all three equations. For the second one $x = \sqrt[3]{-\frac{1}{7}} = -\frac{1}{\sqrt[3]{7}}$, $y = \sqrt[3]{\frac{27}{7}} = \frac{3}{\sqrt[3]{7}}$ and $z = \sqrt[3]{\frac{125}{7}} = \frac{5}{\sqrt[3]{7}}$ (which are all real) and so $xyz = -\frac{15}{7}$. Again, these do satisfy all three equations.

In conclusion $x^3 + y^3 + z^3$ can take the values 16 or $\frac{151}{7}$.

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Hanna Neumann

Sheet 6

Solutions and comments

This programme of the Mentoring Scheme is named after Hanna Neumann (1914–1971).

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Version 1.2, Feb 2021

1. A real number x is such that $\sqrt{x+43}$ and $\sqrt{x+16}$ are both integers. Find the possible values of x.

Solution

We let $a = \sqrt{x+43}$ and $b = \sqrt{x+16}$ where a and b are integers. We now relate a and b:

$$a^2 = x + 43 = (x + 16) + 27 = b^2 + 27,$$

so,

$$27 = a^2 - b^2 = (a - b)(a + b).$$

Now a and b are integers so $a - b$ and $a + b$ are integer factors of 27. Furthermore a and b are non-negative (by the definition of the square root) so $a + b$ is non-negative and at least $a - b$. Finally $a - b = \frac{27}{a+b}$ is non-negative. Thus we have two options: $a + b = 27$ and $a - b = 1$ or $a + b = 9$ and $a - b = 3$.

The first of these gives $a = 14$ and $b = 13$ while the second gives $a = 6$ and $b = 3$. If $b = 13$ then $x = b^2 - 16 = 153$, while if $b = 3$ then $x = b^2 - 16 = -7$.

Both of these work: $\sqrt{153+43} = 14$, $\sqrt{153+16} = 13$, $\sqrt{-7+43} = 6$ and $\sqrt{-7+16} = 3$.

Thus x is -7 or 153.

2. *a)* A digital clock shows each minute (in 24-hour format) in a 24-hour period as a string of 4 digits. How many such strings of digits are there? Explain your answer fully.

b) How many of the strings in part *a)* represent valid times when read in reverse order? Explain your answer fully.

Solution

In what follows it is assumed that midnight is represented by 0000, not 2400.

a) Since there is a one-one correspondence between the number of strings and the number of minutes in 24 hours, the required total is $24 \times 60 = 1440$.

b) First consider what strings representing times look like (if they are read left to right). The first digit can be 0, 1 or 2 and the choices for the second digit depend on this first digit. If the string starts with 0 or 1, then there are 10 choices for the second digit; if the string starts with 2, then there are only 4 (0, 1, 2, 3). The third and fourth digits deal with the minutes: there are 6 choices for the third digit (0, 1, 2, 3, 4, 5) and 10 for the fourth.

Now label the positions A, B, C and D when reading from left to right. Consider what happens when a string is read in reverse (starting in the D position). The D digit is now restricted to be 0, 1 or 2. If it is 0 or 1, are there then 10 choices for the C digit? No, because there are just 6 choices for the C digit when the string is read in the original order. In a reverse reading the B digit is a minutes digit with 6 choices and the A digit is restricted to be 0, 1, or 2 anyway. If the D digit is 2, then the pattern is similar but with different numbers of choices.

The previous paragraph shows that the construction of a valid string that represents a time when read in either order must take account of the end digits. There are four possibilities to consider: 0 or 1 paired with 0 or 1; 0 or 1 paired with 2; 2 paired with 0 or 1; 2 paired with 2.

The numbers of choices are conveniently displayed in a table. Take care when reading the table: the entries in the first two columns are possible digits; in the other columns they give the numbers of choices.

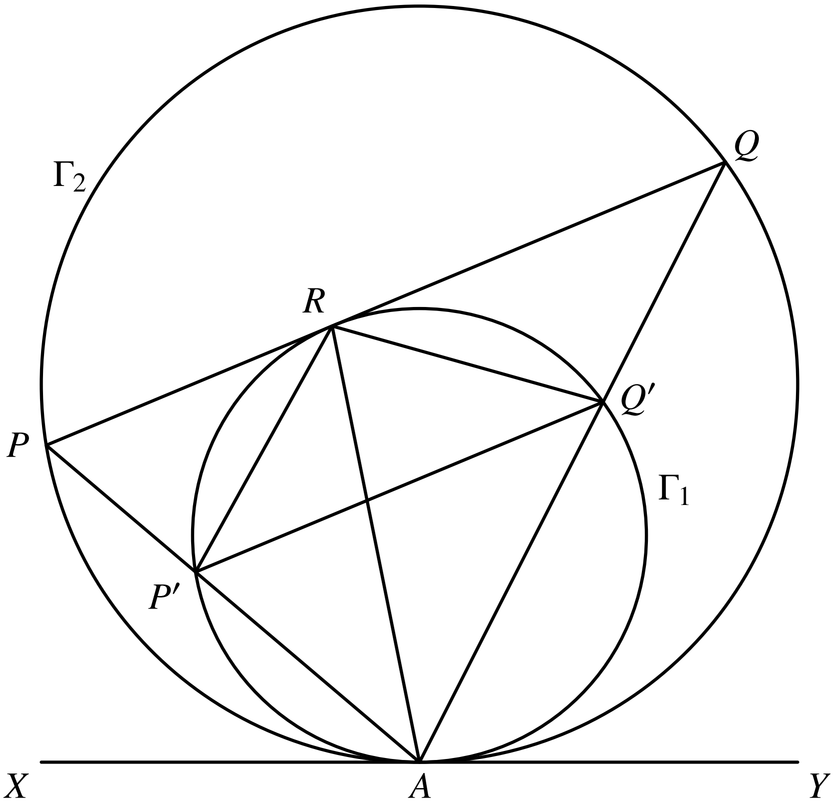
A digit	D digit	A choices	B choices	C choices	D choices	Total
0 or 1	0 or 1	2	6	6	2	144
0 or 1	2	2	6	4	1	48
2	0 or 1	1	4	6	2	48
2	2	1	4	4	1	16

The overall total of valid strings is therefore 144 + 48 + 48 + 16 = 256.

3. Circle Γ_1 lies inside circle Γ_2 and touches it at point A. Chord PQ of Γ_2 is tangent to Γ_1 at R. Show that RA bisects $\measuredangle PAQ$.

Solution

With all these tangents it seems prudent to apply the alternate segment theorem in both circles Γ_1 and Γ_2. To this end let P' be the other intersection of AP with Γ_1 and Q' be the other intersection of AQ with Γ_1 and finally let X and Y lie on the tangent at A as shown.



We are asked to show that $\measuredangle P'AR = \measuredangle RAQ'$. Now, $\measuredangle RAQ' = \measuredangle RP'Q'$ by equal angles in the same segment and $\measuredangle P'AR = \measuredangle P'RP$ by the alternate segment theorem applied to Γ_1. Hence it is enough to show $\measuredangle RP'Q' = \measuredangle P'RP$. We will do this by proving that PQ and $P'Q'$ are parallel lines so that the desired equality holds by equal alternate angles.

We shall obtain this parallel property by using the converse of the corresponding angle result. The requisite equal angles are obtained by using the alternate segment theorem twice. Thus $\measuredangle QPA = \measuredangle QAY$ by angle in the alternate segment applied to Γ_2. But $\measuredangle QAY = \measuredangle Q'P'A$ by angle in the alternate segment applied to Γ_1. Hence $\measuredangle QPA = \measuredangle Q'P'A$ which are the two corresponding angles we require.

4. *a*) In your maths set there are identical triplet brothers called John, James and Julius. John and James always lie but Julius always tells the truth. They are indeed indistinguishable in appearance. You bump into one of them at break and want to find out if he is John, because John has borrowed your maths homework. An extra complication is that any brother responds to a question only if it can be answered by “Yes” or “No”. You may ask just one question. What question would you ask? Justify your answer.

b) Your maths set also contains identical (and indistinguishable) twin sisters, Andrea and Araminta. One always tells the truth and the other always lies but you don’t know which tells the truth. You meet them on the school bus and you want to find out which is Andrea because you have lent her your computer science homework. What Yes / No question would you ask in this case? Justify your answer.

Solution

a) The question that you ask must certainly elicit a distinguishable answer from John. Suppose you go for the simplest option and say to the guy you meet “Are you John?”. Then, if he is John he will answer “No” (he lies); if he is James he will answer “Yes” (he lies); if he is Julius he will answer “No” (he is telling the truth). This provides a distinguishable answer from one brother but you don’t know what significance to attach to it. Thus *in advance* you need to decide what answer you want.

Suppose we adapt the question so that John answers “Yes” and the others “No”. John would certainly answer “Yes” if he were asked if he were James or Julius. So try the question “Are you James?” If you ask James he will untruthfully answer “No”. If you ask Julius he will truthfully answer “No”. Thus a “Yes” answer indicates you are speaking to John; a “No” answer indicates you are not speaking to John.

b) This time you have no knowledge linking a person and her (un)truthfulness, suggesting that the question should involve this aspect of the situation. Suppose you say to a twin “Is Andrea truthful?” If you are speaking to Andrea and she is a truth-teller, then she will answer “Yes”. On the other hand, if Andrea lies, then she will (falsely) claim that she is truthful and therefore say “Yes”.

Now suppose you ask Araminta “Is Andrea truthful?” If Araminta is the truth-teller, then Andrea is the liar and Araminta will say “No”. On the other hand, if Araminta is the liar, then Andrea tells the truth and Araminta will (falsely) say she lies. Thus her reply will once more be “No”. Therefore, irrespective of whether she tells the truth or not, the twin who answers “Yes” is Andrea.

5. Imagine an infinite chessboard where each square contains a positive integer and the integer in each square is the mean of the integers in the four neighbouring squares. Prove that the values in all the squares are equal.

Solution

If the squares all contain positive values, then, of the integers used, there must be a minimum value M, say. Consider a square (not necessarily unique) that contains M. Then M is the mean of the values in the four neighbouring squares. These integers are all greater than or equal to M. Further, by the definition of the mean, their sum is equal to $4M$. If even one of these integers were strictly greater than M, then their sum would be strictly greater than $4M$. Hence the values in the four squares must all be equal, and equal to M. By repeating this argument, we see that the integers in all the squares are equal.

6. Find all integers $n \geqslant 1$ for which there exist n consecutive positive integers whose sum is a prime number.

Solution

The integer 2 is prime so the sequence $\{2\}$ satisfies the case $n = 1$. Note that this example completes the argument. Of course, we could use any prime but one example suffices.

A similar approach works for the case $n = 2$, where the example $\{2, 3\}$ clinches the argument because $2 + 3 = 5$. Note that, if p is any odd prime, then $\frac{1}{2}(p-1) + \frac{1}{2}(p+1) = p$. However, we are not asked to consider this general case and again one numerical example suffices.

Now consider the sum $a + (a + 1) + \ldots + (a + k) = S$, say, where $a \geqslant 1$ and $k \geqslant 2$. We have

$$S = a + (a + 1) + \cdots + (a + k - 1) + (a + k)$$

and

$$S = (a + k) + (a + k - 1) + \cdots + (a + 1) + a.$$

Adding corresponding pairs yields $2S = (k + 1)(2a + k)$. Note that in this product $k + 1 \geqslant 3$ and $2a + k \geqslant 4$.

If k is odd, set $k = 2m + 1$. Since k is odd and greater than or equal to 2, we must have $k \geqslant 3$; this in turn implies that $m \geqslant 1$. Substituting in the expression for $2S$ yields $2S = 2(m+1)(2a+2m+1)$ or $S = (m + 1)(2a + 2m + 1)$. But here $m + 1$ is at least 2 so S cannot be prime.

On the other hand, if k is even, then we can set $k = 2l$. This yields $2S = 2(2l + 1)(a + l)$, implying that $S = (2l + 1)(a + l)$. But $k \geqslant 2$ implies $l \geqslant 1$. Thus both $2l + 1$ and $a + l$ are greater than or equal to 2. We again have a non-trivial factorisation of S and so it cannot be prime.

We conclude that the only values of n for which the result holds are $n = 1$ and $n = 2$.

Remark: If you have met the idea of an AP (an arithmetic progression), then you will recognise the argument in the last part of the proof as the standard method for summing an AP.

7. Prove that if $2^n - 1$ is prime then n must be prime. What about the converse result: can you find a prime n such that $2^n - 1$ is composite?

Solution

Here is an example of a question where it pays to turn the argument round. If we assume that $2^n - 1$ is prime, then where do we go from there? If, on the other hand, a composite integer is the starting point, then at least it is possible to factorise it. However, in this case it is important to be careful about the direction of the logical implication. We need to prove that if n is composite, then so is $2^n - 1$. (Make sure you understand why this is equivalent to the desired result.)

To get a small detail out of the way, note that it makes sense to consider $n > 1$ only. If $n \geqslant 2$ is composite, then it can be factorised as $n = ab$, where $a > 1$ and $b > 1$. Consider

$$2^n - 1 = 2^{ab} - 1 = ((2^a)^b - 1).$$

To tame the notation here, set $2^a = x$. Note $a > 1$ implies $x \geqslant 4$. Then $2^n - 1 = x^b - 1$ and we can now employ the following extremely useful factorisation.

$$2^n - 1 = x^b - 1 = (x - 1)(x^{b-1} + x^{b-2} + \cdots + x + 1).$$

This factorisation needs checking to make sure it is not trivial. First note that $x - 1 \geqslant 3$. Again, since $b > 1$, then $b - 1 \geqslant 1$. This implies that the term $x + 1 \geqslant 5$ always appears in the second bracket. Further, it is reassuring to note that the smallest value of n considered must be $n = 4$ and in that case the given factorisation is $2^4 - 1 = ((2^2)^2 - 1) = (2^2 - 1)(2^2 + 1) = 3 \times 5$, as expected. To recapitulate, we have shown that n composite implies $2^n - 1$ composite. But this is equivalent to the statement that $2^n - 1$ prime implies n prime.

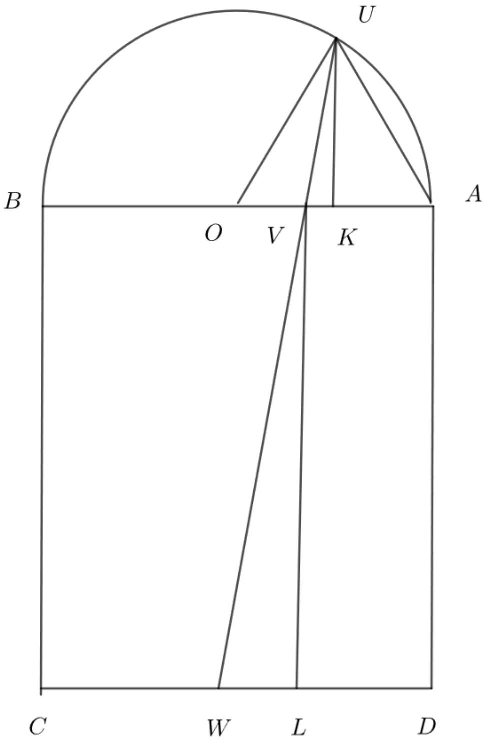
For the example, we do not have far to look. Though $2^n - 1$ is prime for $n = 2$, 3, 5 and 7, it turns out that $2^{11} - 1 = 2047 = 23 \times 89$.

Remark: If you have met the idea of a GP (a geometric progression), then you may want to consider how the factorisation of $x^b - 1$ *is related to the formula for the sum of a GP.*

8. The rectangle $ABCD$ and a semicircle with diameter AB are coplanar and their interiors do not overlap. Let $\mathcal{R}$ denote the region enclosed by the rectangle and the semicircle. Line ℓ meets the semicircle in U, the segment AB in V and the segment CD in W, where the points U, V and W are distinct. Also the line ℓ divides the region $\mathcal{R}$ into two regions $\mathcal{S}$ and $\mathcal{T}$. Suppose that $\mathcal{T}$ is the region bounded by the arc AU and the line segments UW, WD and DA in order. The regions $\mathcal{S}$ and $\mathcal{T}$ have areas in the ratio 2:1. Suppose further that the line segment $AU = 6$, $AV = 4$ and $VB = 8$. Find the length DA.

Solution

The configuration is shown in the diagram below. In this diagram K is the foot of the perpendicular from U onto AB and L is the foot of the perpendicular from V onto CD.



The smaller part of $\mathcal{R}$, one third of the whole, is the region $\mathcal{T}$ bounded by the arc AU and the segments UW, WD and DA in order. Because $AV = 4$ and $VB = 8$, the diameter AB of the semicircle has length 12 and so its radius is 6. Denote the midpoint of AB by O; this is the centre of the semicircle and, since $AV = 4$, we see that $OV = 2$.

It is now crucial to note that triangle AUO is equilateral: this is because $UO = OA = 6$ (equal radii) and $AU = 6$ (given). This observation has the following consequences.

- $\angle OAU = 60^\circ$ so $KU = 6\sin 60^\circ = 3\sqrt{3}$.
- $\angle UOA = 60^\circ$ so the area of the sector UOA is $\frac{1}{3}C$, where C is the area of the semicircle.
- $KA = 6\cos 60^\circ = 3$ so $VK = 4 - 3 = 1$.

The upper part of the region $\mathcal{T}$ is the sector UOA with the triangle UOV removed. Thus

$$\text{area of upper part} = \frac{1}{3}C - \frac{1}{2} \times 6 \times 2\sin 60^\circ = \frac{1}{3}C - 3\sqrt{3}. \tag{1}$$

The lower part of the region $\mathcal{T}$ is the trapezium $AVWD$. If we set $LV = h$, then

$$\text{area of lower part} = \frac{1}{2}(4 + WD)h. \tag{2}$$

We now need to find WD. However, there is an obvious pair of similar triangles: VWL and UVK. They give us the equal ratios

$$\frac{WL}{LV} = \frac{VK}{KU} \text{ and so } \frac{WL}{h} = \frac{VK}{KU} \text{ and hence, using the observations above, } WL = \frac{1}{3\sqrt{3}}h. \tag{3}$$

Now, since $WD = WL + LD = WL + 4$, we have $WD = \frac{1}{3\sqrt{3}}h + 4$. Substituting in (2) gives

$$\text{area of lower part} = \frac{1}{2}\left(4 + \left(4 + \frac{1}{3\sqrt{3}}h\right)\right)h = 4h + \frac{1}{6\sqrt{3}}h^2. \tag{4}$$

Combining (1) and (4), we now have

$$\text{area of } \mathcal{T} = \frac{1}{3}C - 3\sqrt{3} + 4h + \frac{1}{6\sqrt{3}}h^2 \tag{5}$$

This area is one third of the area of $\mathcal{R}$, which is $C + 12h$. Hence, using (5),

$$\frac{1}{3}C - 3\sqrt{3} + 4h + \frac{1}{6\sqrt{3}}h^2 = \frac{1}{3}C + 4h. \tag{6}$$

Simplifying yields $h^2 = 6\sqrt{3} \times 3\sqrt{3} = 6 \times 9$, so $h = LV = 3\sqrt{6}$. However $DA = LV$ and so the final answer is $DA = 3\sqrt{6}$.

Remark: It is possible to construct a more geometrical proof that uses the fact that the square root of an area scale factor gives the corresponding linear scale factor. However, in this case it is harder to select appropriate similar triangles.

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Hanna Neumann

Sheet 7

Solutions and comments

This programme of the Mentoring Scheme is named after Hanna Neumann (1914–1971).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Neumann_Hanna.html for more information, or search “Hanna Neumann mathematician” to find a wider range of sources.

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1. Show how to write the number 1066 using each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 once and once only. You may use any mathematical operation that you wish. Can you repeat the process if the zero is omitted?

Solution

There must be all sorts of ways of answering this question. One way, perhaps surprisingly, is to start from the prime factorisation of 1066. This is $1066 = 2 \times 533 = 2 \times 13 \times 41$. The largest factor 41 is likely to be the most awkward but it can be expressed as $41 = 8 \times 5 + 1$. This can be doubled by multiplying by $\frac{4}{2}$. (This uses up two digits rather than one). Then $13 = 6 + 7$ and the remaining non-zero factors can be bundled up with 0 to obtain the final expression of

$$1066 = \frac{4}{2}(6+7)\big(8 \times 5 + 1\big) + 0 \times 3 \times 9.$$

When zero is omitted this can be adapted by replacing it by

$$1066 = \frac{4}{2}\big(9 + 3 + 7 - 6\big)\big(8 \times 5 + 1\big).$$

2. The French mathematician Évariste Galois was killed, aged only 21, in a duel in 1832. Which of the integers 3663! and 1832^{3663} is greater?

Solution

The two numbers we are comparing are products of 3663 integers. Since this is far too much to deal with in one go, a sensible approach is to split up each of the products so we are dealing with only a small number of terms at once. We aim to pair up terms from 3663! with terms from 1832^{3663} in such a way that those from 3663! are either always smaller or always larger than those from 1832^{3663}. It will then be easy to compare the two numbers and this is what we shall try to do. Now

$$3663! = 3663 \times 3662 \times \cdots \times 1832 \times \cdots \times 2 \times 1.$$

We would like to compare this with a product of 3663 1832's so it seems sensible to pair up large terms with small terms:

$$3663! = (3663 \times 1) \times (3662 \times 2) \times \cdots \times (1833 \times 1831) \times 1832.$$

We would now like to compare each bracketed pair with 1832^2. Each bracket is of the form $a(3664 - a)$, where $a = 1, 2, \ldots, 1831$. Completing the square gives

$$a(3664 - a) = \underline{\quad\quad}^2 - (\underline{\quad} - a)^2 < 1832^2.$$

Thus $3663! < (1832^2)^{1831} \times 1832 = 1832^{3662} \times 1832 = 1832^{3663}$.

Remark: Alternatively the inequalities can be derived from the AM-GM inequality. For this see, for example, Question 4 of Sheet 4.

3. *a*) Suppose that a positive integer N has the prime factorisation $N = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$. Explain why the number of divisors of N is $(k_1 + 1)(k_2 + 1) \cdots (k_n + 1)$.

b) In 1485 Henry Tudor defeated Richard III on Bosforth Field. He thereby inaugurated the Tudor dynasty and ushered in 118 tumultuous and colourful years of English history. Write down the number of divisors of 1485.

c) Suppose you made a list of the divisors of 1485^2 and then randomly selected two distinct divisors from this list. What would be the probability that exactly one of the selected divisors was a perfect square?

Solution

a) Any divisor of N must have a prime factorisation of the form $p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, where, for each subscript i, $0 \leqslant r_i \leqslant k_i$. In other words, the power r_i can take $k_i + 1$ different values. It follows that the factor $p_i^{r_i}$ can take $k_i + 1$ different values. Thus when one is composing a divisor of N, there are $k_1 + 1$ choices for the power of p_1, $k_2 + 1$ choices for the power of p_2, and so on, up to $k_n + 1$ choices for the power of p_n. Overall the number of distinct divisors is therefore $(k_1 + 1)(k_2 + 1) \cdots (k_n + 1)$, as required.

b) The prime factorisation of 1485 is $1485 = 3^2 \times 165 = 3^3 \times 5 \times 11$, so the required number of divisors is $4 \times 2 \times 2 = 16$.

c) Since $1485^2 = 3^6 \times 5^2 \times 11^2$, the number of divisors of 1485^2 is $7 \times 3 \times 3 = 63$. A divisor of 1485^2 has the form $3^r \times 5^s \times 11^t$, where $0 \leqslant r \leqslant 6$ and each of s and t can take one of the values 0, 1, 2. Such a divisor is a perfect square precisely when r, s and t are all even. Therefore r can take one of the values 0, 2, 4, 6 and each of s and t can be either 0 or 2. Hence the number of divisors which are perfect squares is $4 \times 2 \times 2 = 16$. Note that the number of non-square divisors divisors is $63 - 16 = 47$.

Now suppose your random choice from the list turns out to be a divisor that is a perfect square, followed by a non-square. The probability of this occurring is

$$\frac{16}{63} \times \frac{47}{62} = \frac{8 \times 47}{31 \times 63}.$$

However, another outcome that provides only one square occurs if the random selection picks a non-square divisor, followed by a square divisor. Thus

$$\text{the probability of obtaining exactly one square divisor} = 2 \times \frac{8 \times 47}{31 \times 63} = \frac{16 \times 47}{31 \times 63}.$$

If you are familiar with the idea of a ***binomial coefficient***, there is a slightly quicker way of obtaining this result. The number of ways of choosing two divisors is

$$\binom{63}{2} = \frac{63 \times 62}{2 \times 1} = 31 \times 63.$$

The number of ways of choosing a divisor that is a perfect square is 16; the number of ways of choosing a non-square divisor is 47. Hence

$$\text{the probability of obtaining exactly one square divisor} = \frac{16 \times 47}{31 \times 63}, \quad \text{as before.}$$

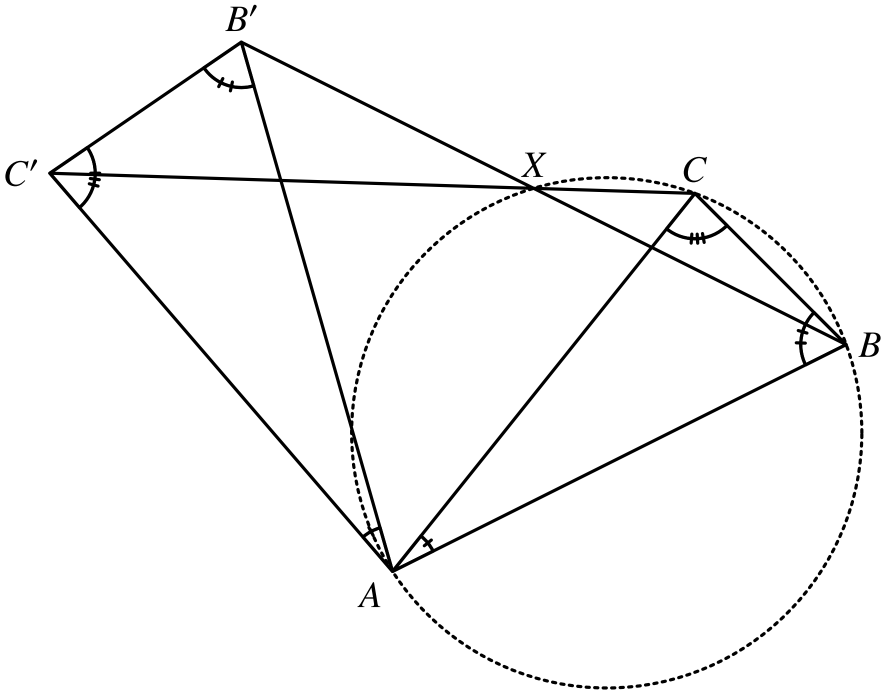
Remark for newcomers: There is a wealth of information available about binomial coefficients and their properties: you may want to consult your mentor about a good place to start.

4. Let points A, B, C, B' and C' be such that ray AB' is between rays AC and AC', ray AC is between rays AB and AB', and triangle ABC is similar to triangle $AB'C'$ (with the similarity in that order). Let lines BB' and CC' meet at X. Show that X lies on the circle through A, B and C and on the circle through A, B' and C'.

SOLUTION

Solution 1: If we show that X lies on the circle through A, B and C then it must automatically lie on circle through A, B' and C' by symmetry (swapping B for B' and C for C' does not change the position of X).

We are aiming to show that $ABCX$ is a cyclic quadrilateral so it is enough to show that $\angle ABX = \angle ACX$ i.e. that $\angle ABB' = \angle ACC'$. Note that in the diagram below the angles marked as equal come from the similarity of triangles ABC and $A'B'C'$.



We do know that $\angle B'AB = \angle B'AC + \angle CAB = \angle B'AC + \angle C'AB' = \angle C'AC$ so showing that $\angle ABB' = \angle ACC'$ is the same as showing that triangles ABB' and ACC' are similar.

We are told that triangles ABC and $AB'C'$ are similar so we know

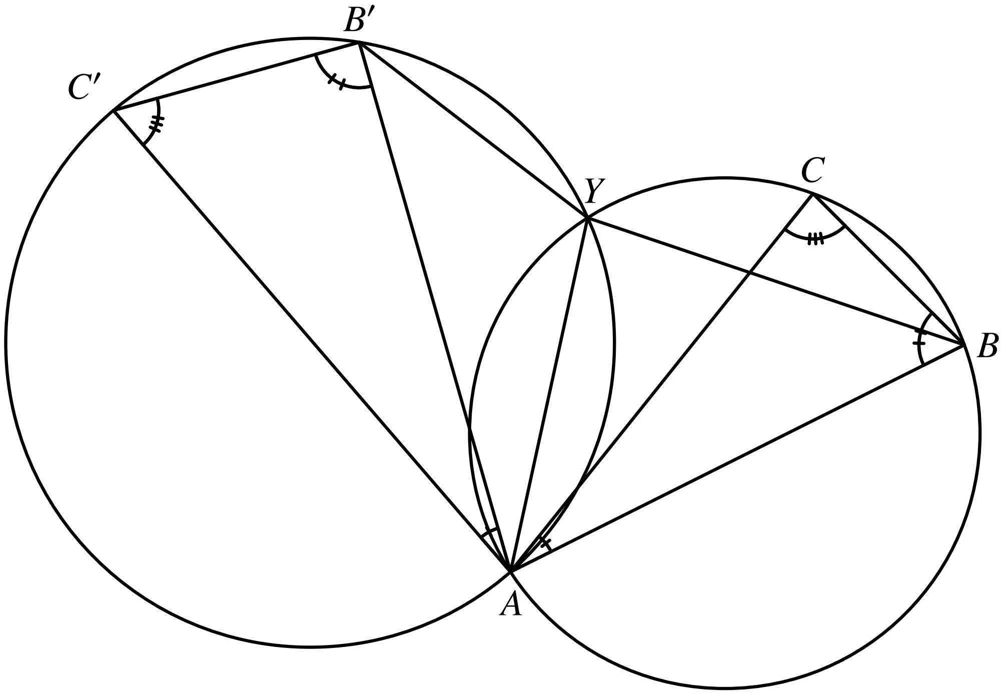
$$\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'}. \tag{1}$$

We wish to deduce that ABB' and ACC' are similar. There are various ways to prove similarity (see Sheet 3 Question 4) and here the SAS criterion (two sides in ratio and the included angle equal) is perfect: we already have shown $\angle B'AB = \angle C'AC$ and from (1) we have $\frac{AB}{AC} = \frac{AB'}{AC'}$.

Solution 2: This problem can also be solved by working backwards: we let Y be the point (other than A) where circles ABC and $AB'C'$ meet. It is then enough to show that Y lies on BB' (and so also on CC' by symmetry) as then Y must be X and so X lies on the two circles we want.

This gives us a different (but equivalent) problem and allows us to start with different information (we can use the fact that $ABCY$ and $AB'C'Y$ are cyclic quadrilaterals) and alter what we need to prove (that B, Y and B' are collinear). When given a geometry problem it is worth considering whether another approach like this might be helpful.

The diagram below has been drawn badly to ensure that B, Y and B' do not appear to lie on the same straight line. This helps us not to assume they do in the proof!



Now $\measuredangle BYA = \measuredangle BCA$ using angles in the same segment and $\measuredangle BCA = \measuredangle B'C'A$ using the similar triangles. Also $\measuredangle AYB' + \measuredangle AC'B' = 180°$ as they are opposite angles in a cyclic quadrilateral. Thus $\measuredangle AYB' + \measuredangle BYA = 180°$ so BYB' is a straight line, as required.

5. *a*) Find the maximum value of xy^3z^5 where x, y and z are positive real numbers with $x + y + z = 3$

The *harmonic mean* (HM) of *positive* real numbers $a_1, a_2, \dots, a_n$ is

$$\frac{1}{\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}},$$

that is, the reciprocal of the mean of the reciprocals.

b) Let $a_1, a_2, \dots, a_n$ be positive real numbers. Prove that their geometric mean is at least their harmonic mean and determine when equality occurs.

c) Show that for any positive real numbers x, y and z with $x + y + z = 1$ we have

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} \geqslant \frac{9}{2}.$$

When does equality occur?

d) Show that for any positive real numbers x, y and z we have

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geqslant \frac{3}{2}.$$

SOLUTION

a) We wish to find the maximum value of a product and we are given the value of a sum so it seems sensible to try to use AM-GM. As in part *c)* of Question 5 on Sheet 5 we want to split up the sum so that when we apply AM-GM we get the powers of 3 and 5: we write $x + y + z$ as $x + \frac{y}{3} + \frac{y}{3} + \frac{y}{3} + \frac{z}{5} + \frac{z}{5} + \frac{z}{5} + \frac{z}{5} + \frac{z}{5}$ and apply AM-GM:

$$\begin{aligned}\tfrac{1}{3} = \tfrac{1}{9}\left(x + \tfrac{y}{3} + \tfrac{y}{3} + \tfrac{y}{3} + \tfrac{z}{5} + \tfrac{z}{5} + \tfrac{z}{5} + \tfrac{z}{5} + \tfrac{z}{5}\right) &\geqslant \sqrt[9]{x \times \tfrac{y}{3} \times \tfrac{y}{3} \times \tfrac{y}{3} \times \tfrac{z}{5} \times \tfrac{z}{5} \times \tfrac{z}{5} \times \tfrac{z}{5} \times \tfrac{z}{5}} \\ &= \sqrt[9]{\tfrac{xy^3z^5}{3^35^5}},\end{aligned}$$

so

$$xy^3z^5 \leqslant \frac{3^35^5}{3^9} = \frac{5^5}{3^6}.$$

Now we had better check that there are x, y and z satisfying $x + y + z = 3$ and $xy^3z^5 = \frac{5^5}{3^6}$ as otherwise we only know that the maximum value of xy^3z^5 is at most $\frac{5^5}{3^6}$ and cannot be sure that it is possible to obtain $\frac{5^5}{3^6}$.

To get this maximum value we would need equality in AM-GM so we would need $x = \frac{y}{3} = \frac{z}{5}$ so $y = 3x$ and $z = 5x$. Then $3 = x + y + z = 9x$ so $x = \frac{1}{3}$, $y = 3x = 1$ and $z = 5x = \frac{5}{3}$. For this x, y and z we do have $x + y + z = 3$ and

$$xy^3z^5 = \frac{1}{3} \times 1 \times \frac{5^5}{3^5} = \frac{5^5}{3^6},$$

so $\frac{5^5}{3^6}$ is indeed the maximum value.

b) We are asked to show that for $a_1, a_2, \ldots, a_n$ positive real numbers we have

$$\sqrt[n]{a_1 a_2 \cdots a_n} \geqslant \frac{1}{\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)}.$$

Rearranging to get rid of the reciprocal of a sum of reciprocals we see that what we need to prove is equivalent to

$$\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geqslant \frac{1}{\sqrt[n]{a_1 a_2 \cdots a_n}}.$$

Now the left-hand side is the arithmetic mean of $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ and right-hand side is their geometric mean:

$$\frac{1}{\sqrt[n]{a_1 a_2 \cdots a_n}} = \sqrt[n]{\frac{1}{a_1} \times \frac{1}{a_2} \times \cdots \times \frac{1}{a_n}},$$

so we are done by the AM-GM inequality. We have equality if and only if we have equality when we apply AM-GM which happens if and only if $\frac{1}{a_1} = \frac{1}{a_2} = \cdots = \frac{1}{a_n}$. And this occurs if and only if all the a_i are equal.

Remark: We now know that for positive reals AM $\geqslant$ GM $\geqslant$ HM with equality if and only if all the variables are equal. In fact this can be extended further: if you are interested you might want to look up the quadratic mean-arithmetic mean inequality. These are all special cases of generalised mean / power mean inequalities.

c) The left-hand side of this inequality is a sum of reciprocals so looks a bit like a harmonic mean. We are given the value of a sum so it makes sense to try and use an inequality relating the arithmetic mean and harmonic mean: by part *b)* (see the Remark) $AM \geqslant HM$. Applying this to $1 - x$, $1 - y$ and $1 - z$ we get:

$$\frac{(1-x) + (1-y) + (1-z)}{3} \geqslant \frac{3}{\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}}.$$

The left-hand side of this equals $\frac{3-(x+y+z)}{3} = \frac{2}{3}$ and so

$$\frac{2}{3} \geqslant \frac{3}{\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z}},$$

multiplying up gives us the result we are looking for.

Now for equality we need equality in both AM-GM and GM-HM which we know occurs if and only if $1 - x = 1 - y = 1 - z$ or $x = y = z$. But $x + y + z = 1$ so we have equality exactly if $x = y = z = \frac{1}{3}$.

d) ***Solution 1:*** If we take the inequality from part *c)* and replace every 1 by $x + y + z$ we see that we have shown the for positive reals x, y and z satisfying $x + y + z = 1$:

$$\frac{x+y+z}{y+z} + \frac{x+y+z}{x+z} + \frac{x+y+z}{x+y} \geqslant \frac{9}{2}.$$

The left-hand side is $\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} + 3$ so subtracting 3 from both sides gives us exactly what we want:

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geqslant \frac{3}{2}.$$

So we have shown the inequality (which is called *Nesbitt's inequality*) provided $x + y + z = 1$.

What if $x + y + z$ is not 1? We could scale x, y and z to achieve this: suppose $x + y + z = \lambda$. Let $x' = \frac{x}{\lambda}$, $y' = \frac{y}{\lambda}$ and $z' = \frac{z}{\lambda}$ so that $x' + y' + z' = 1$, $x = \lambda x'$, $y = \lambda y'$ and $z = \lambda z'$.

Now

$$\begin{aligned}\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} &= \frac{\lambda x'}{\lambda y' + \lambda z'} + \frac{\lambda y'}{\lambda z' + \lambda x'} + \frac{\lambda z'}{\lambda x' + \lambda y'} \\ &= \frac{x'}{y'+z'} + \frac{y'}{z'+x'} + \frac{z'}{x'+y'}.\end{aligned}$$

But, as $x' + y' + z' = 1$, we know

$$\frac{x'}{y'+z'} + \frac{y'}{z'+x'} + \frac{z'}{x'+y'} \geqslant \frac{3}{2},$$

and so we are done!

Solution 2: A much more messy method is to clear all the denominators and multiply out: we need to show

$$2x(x+y)(x+z) + 2y(y+z)(y+x) + 2z(z+x)(z+y) \geqslant 3(x+y)(y+z)(z+x),$$

which upon multiplying out and cancelling becomes

$$2x^3 + 2y^3 + 2z^3 \geqslant x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2.$$

Now although we have three variables x, y and z each term contains at most 2 variables so it makes sense to split up the inequality into three parts: if we can show

$$x^3 + y^3 \geqslant x^2y + xy^2, \tag{$*$}$$

then we similarly have $y^3 + z^3 \geqslant y^2z + yz^2$ and $z^3 + x^3 \geqslant z^2x + zx^2$ — summing these three up gives what we want.

There are a few ways to prove $(*)$: the simplest is to pull everything onto the left-hand side and notice that $x^3 - x^2y - xy^2 + y^3 = (x-y)\left(x^2 - y^2\right) = (x+y)(x-y)^2$ which is certainly non-negative. Another is to use AM-GM:

$$\begin{aligned}\frac{x^3 + x^3 + y^3}{3} &\geqslant \sqrt[3]{x^3 \times x^3 \times y^3} = x^2y, \\ \frac{x^3 + y^3 + y^3}{3} &\geqslant \sqrt[3]{x^3 \times y^3 \times y^3} = xy^2,\end{aligned}$$

then adding terms of this sort together gives $(*)$.

6. Container A has 3 litres of syrup in it. Container B has n litres of water in it. Container C is empty. There are no markings on the containers but we have some means of knowing when two containers have the same amount of liquid in them. (For example, they might be identical glass cylinders, enabling us to check that the level of liquid is the same in two of them.) We can perform any sequence of the following operations.

- Pour away the entire amount in any container.
- Pour the entire amount in one container into another container.
- Pour liquid from one container into another container until the amount in the recipient is equal to the amount in the third container.

a) Now assume that $n = 20$. Explain how we can obtain 10 litres of liquid in one container that is 30% syrup and 70% water.

b) Determine all the values of n for which the task in part *a)* is possible.

Solution

a) Two preliminary remarks about the amount of water in B may help to point the way forward to an overall strategy. First, there is a lot of it and we need to finish with only 10 litres of liquid. Hence we may be able to remove a lot of water without jeopardising the end of the process. This also suggests that the value $n = 20$ is not particularly special and we may be able to devise a procedure that can be extended to other values of n.

The syrup in A can be used to measure 3 litres of water. Hence the water in B may be reduced by 3 litres at a time by the following procedure: pour water from B into C until C contains 3 litres; pour away the water in C. Continue until: A contains 3 litres of syrup; B contains 8 litres of water; C is empty. Note that we cannot remove another 3 litres of water from B and pour it away. (Why?) However, still using A as a measure, we can pour 3 litres of water from B into C and leave it there. Note that we now have 11 litres of liquid altogether, so we need to end up with 1 litre of water in either B or C.

The end of the process can conveniently be summarised in the following table, where W denotes water, S denotes syrup and all quantities are in litres.

Action	Amount in A	Amount in B	Amount in C
(Initial state)	3S	8W	0
$B \to C$ until $C = A$	3S	5W	3W
$A \to C$ until $C = B$	1S	5W	3W and 2S
$B \to A$ until $A = C$	1S and 4W	1W	3W and 2S
$C \to A$	3S and 7W	1W	0

b) Suppose we move on from *a)* and consider $n = 21$. Again we can start by removing water from B, 3 litres at a time. (Note that 3 litres is the only amount we can measure.) This time the number of litres in any container is a multiple of 3. Even if we start measuring against B at the same stage as before, B has 6 litres of water in it. Thus we can never reach 10 because 10 is not a multiple of 3. Hence the task is not possible if $n \equiv 0 \bmod 3$.

Next suppose $n \equiv 1 \bmod 3$ and consider small values first.
If $n = 1$ or $n = 4$, then there is not enough water to complete the task.
If $n = 7$, then we simply pour all the water in B into A.

If $n \geqslant 10$, then we can reduce the amount of water 3 litres at a time until B contains 7 litres. These 7 litres then go into A.

Finally let $n \equiv 2 \bmod 3$ and note that this is true of $n = 20$.
If $n = 2$ or $n = 5$, then there is not enough enough water to complete the task.
If $n = 8$, then we have the initial state in the first row of the table above and can proceed as shown there.
If $n \geqslant 8$, then we can reduce the amount of water 3 litres at a time until B contains 8 litres. Again the table shows how the task may be completed.

Thus the task is always possible unless $n = 1, 2, 4, 5$ or $n \equiv 0 \bmod 3$.

7. *a*) A tournament has n contestants. Each contestant plays exactly one game against every other contestant. Explain why the total number of games is $\frac{1}{2}n(n-1)$.

b) In a chess tournament every contestant is supposed to play exactly one game against every other contestant. However, contestant A withdrew from the tournament after playing only ten games and contestant B withdrew after just one game. In all 55 games were played.
Did A and B play each other?

[Mathematical Olympiad for Girls 2015 Question 4]

Solution

a) Each of the n contestants plays $n - 1$ others. However, the game in which X plays Y appears in X's list of games and also in Y's. Hence the total number of games played is $\frac{1}{2}n(n-1)$.
Remark: If you are familiar with binomial coefficients (mentioned above in Question 3), then note that $\frac{1}{2}n(n-1) = \binom{n}{2}$*, the number of ways of choosing 2 contestants from n.*

b) We could look at various possible scenarios but the simplest approach is to consider what happens if A plays B and then what happens if A does not play B. Suppose that there are m contestants other than A and B. The number of games played by these m contestants against each other is $\frac{1}{2}m(m-1)$. There are 10 additional games played by A; we now have to consider two different possibilities.

- Case 1: A did not play B. Then B played one game against one of the other m contestants. The total number of games played was therefore

$$\frac{1}{2}m(m-1) + 10 + 1 = 55.$$

This is $m^2 - m - 88 = 0$ which, by inspection, has no integer solutions. Thus assuming that A did not play B leads to a contradiction.

- Case 2: A did play B. Then this game was on A's list and is the one game that B played, In this case no further game needs to be added. The total number of games played was therefore

$$\frac{1}{2}m(m-1) + 10 = 55.$$

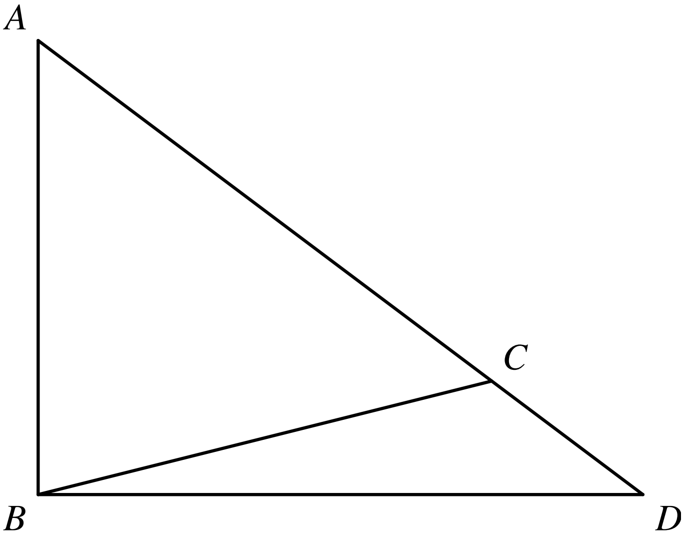
This is $m^2 - m - 90 = 0$. This factorises as $(m-10)(m+9) = 0$, which has the positive integer root $m = 10$. We conclude that A played B and that overall there were 12 contestants.

8. *A trigonometrical exploration:* You may have already met the addition formulae for sin and cos. Even if you have not, this question is designed to show that they can be derived from very simple geometrical and trigonometrical results.

a) By using a suitable right-angled triangle, establish the result $\cos^2 x + \sin^2 x = 1$. (Note that here $\cos^2 x$ means $(\cos x)^2$ and similarly for $\sin^2 x$.)

b) Does this result also hold when x is obtuse?

c) In the diagram below the side AC of the triangle ABC is extended to meet at D the line perpendicular to AB at B. (Thus $\measuredangle ABD$ is a right angle.) Though, strictly speaking, A, B and C denote the vertices of triangle ABC, in what follows we shall use them to denote the ***angles*** of that triangle.



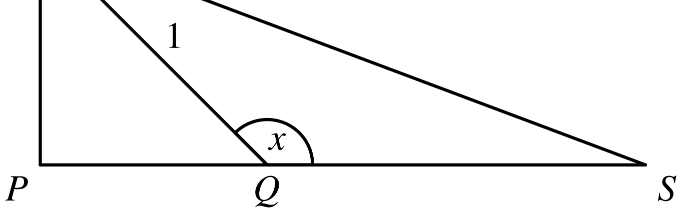
Use the diagram to find expressions for $\sin(A + B)$ and $\cos(A + B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$.

d) Comment on any drawbacks in this method of establishing the addition formulae.

Solution

a) A right-angled triangle PQR with its right angle at P will suffice. Put $\measuredangle PQR = x$. If the hypotenuse QR has length 1, then $PQ = \cos x$ and $RP = \sin x$. The desired result then follows from Pythagoras' theorem.

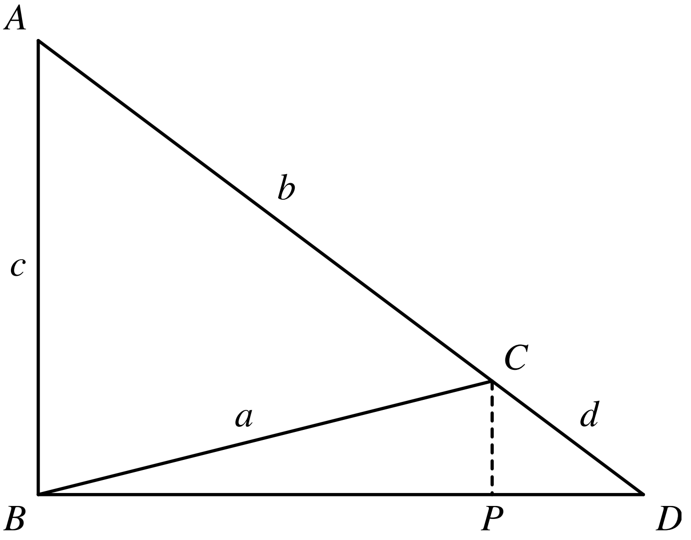
b) Now consider the diagram below: here the angle x is obtuse.



Again the triangle PQR is right-angled with the right angle at P. Also PQS is a straight line and $QR = 1$. By angles on a straight line, $\angle PQR = 180^\circ - x$. In triangle PQR we have $PQ = \cos(180^\circ - x)$ and $RP = \sin(180^\circ - x)$. Now consider the graphs of cos and sin on the interval 0° to 360°. These show that $\cos(180^\circ - x) = -\cos x$ and $\sin(180^\circ - x) = \sin x$. It follows that $PQ = -\cos x$ and $RP = \sin x$. (Note that this is not a nonsensical statement as $\cos x < 0$.) By Pythagoras' theorem we now have

$$PQ^2 + RP^2 = (-\cos x)^2 + (\sin x)^2 = \cos^2 x + \sin^2 x = 1.$$

c) Now we return to the diagram in the question, to which has been added the perpendicular from C onto BD: the foot of this perpendicular is labelled P.



Because $\angle ABD$ is a right angle, CP is parallel to AB. The angle properties of parallel lines now give $\angle DCP = \angle A$ and $\angle PCB = \angle B$. Observe that $\angle DCB = A + B$. It also follows that

$$CP = d \cos A = a \cos B. \tag{1}$$

Again we have

$$BD = BP + PD = d \sin A + a \sin B. \tag{2}$$

Substituting from (1) into (2) now yields

$$BD = \frac{a \cos B}{\cos A} \sin A + a \sin B = \frac{a(\sin A \cos B + \cos A \sin B)}{\cos A}. \tag{3}$$

However, by applying the sine rule to triangle CBD, we obtain

$$\frac{BD}{\sin(A + B)} = \frac{a}{\sin(90^\circ - A)}$$

It now follows, using $\sin(90^\circ - A) = \cos A$, that

$$BD = \frac{a \sin(A + B)}{\cos A}. \tag{4}$$

(Make sure you understand why $\sin(90^\circ - A) = \cos A$.) Now it is possible to equate the two expression for BD in (3) and (4) to obtain

$$\frac{a(\sin A \cos B + \cos A \sin B)}{\cos A} = \frac{a \sin(A + B)}{\cos A}. \tag{5}$$

Cancelling $\dfrac{a}{\cos A}$,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B. \tag{6}$$

The cosine rule is now applied to triangle CBD. This gives $BD^2 = a^2 + d^2 - 2ad\cos(A+B)$. If we substitute from (1), then this expression for BD^2 becomes

$$a^2+\left(\frac{a\cos B}{\cos A}\right)^2-\frac{2a^2\cos B\cos(A+B)}{\cos A} = \left(\frac{a^2}{\cos^2 A}\right)\big(\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B)\big). \tag{7}$$

But from (3) we have an alternative expression for BD^2, namely

$$\left(\frac{a^2}{\cos^2 A}\right)\big(\sin A\cos B + \cos A\sin B\big)^2. \tag{8}$$

Equating the expressions in (7) and (8) and cancelling $\dfrac{a^2}{\cos^2 A}$ produces

$$\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B) = \sin^2 A\cos^2 B+2\cos A\cos B\sin A\sin B+\cos^2 A\sin^2 B. \tag{9}$$

On rearranging this becomes

$$\cos^2 A(1-\sin^2 B) + \cos^2 B(1-\sin^2 A) = 2\cos A\cos B\big(\cos(A+B) + \sin A\sin B\big). \tag{10}$$

Now use the result of part *a)* to replace $(1-\sin^2 B)$ and $(1-\sin^2 A)$ by $\cos^2 B$ and $\cos^2 A$ respectively. Then (10) reduces to

$$2(\cos A\cos B)^2 = 2\cos A\cos B\big(\cos(A+B) + \sin A\sin B\big). \tag{11}$$

Cancelling $2\cos A\cos B$ and rearranging then yields

$$\cos(A+B) = \cos A\cos B - \sin A\sin B. \tag{12}$$

The desired expressions are those in (6) and (12).

d) One drawback is that we have dealt only with angles that might appear in a triangle. This leaves open the question of whether the rules hold for angles of any size. They do but more sophisticated methods are then needed to establish this and you may have met some of these already. There has also been a lot of dividing by and cancelling expressions that might be zero. Strictly speaking these cases should have been examined separately. However, as stated initially, the aim of this question has been to base the exploration of the addition formulae on as simple a foundation as possible.

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Hanna Neumann

Sheet 8

Solutions and comments

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1. The nth triangular number T_n is defined to be the sum of the first n positive integers. Prove that every triangular number greater than 1 is the sum of a square number and two triangular numbers. Is this expression necessarily unique?

Solution

A basic result is that $T_n = \frac{1}{2}n(n+1)$. (Make sure you understand why this is so. If necessary, revisit the argument in Solution 6 of Sheet 6.) You may have a sudden insight and see how this expression could be broken up but what are you to do without this sort of inspiration? An alternative is to examine some small cases and try to express 3, 6, 10, . . . in the required form. An interesting pair consists of $T_6 = 21 = 9 + 6 + 6$ and $T_7 = 28 = 16 + 6 + 6$. These can be rewritten as $T_6 = 3^2 + T_3 + T_3$ and $T_7 = 4^2 + T_3 + T_3$. If $n = 3$, then we see $T_6 = T_{2n} = n^2 + T_n + T_n$. Furthermore, $7 = 2n + 1$ and we have $T_{2n+1} = (n+1)^2 + T_n + T_n$. This suggests, firstly, that we examine odd and even integers separately and, secondly, that we see if the pattern holds in general.

For even integers, we have $T_{2n} = \frac{1}{2} \times 2n(2n+1) = n(2n+1)$. However,

$$n^2 + T_n + T_n = n^2 + 2 \times \frac{1}{2}n(n+1) = n(n+n+1) = n(2n+1).$$

Thus the conjectured result holds for even integers. The line above also shows why $T_n = 1$ is excluded. If $2n + 1 = 1$, then $n = 0$ and T_0 is not defined.

Moving on to odd integers greater than 1, we have $T_{2n+1} = \frac{1}{2} \times (2n+1)(2n+2) = (n+1)(2n+1)$. However,

$$(n+1)^2 + T_n + T_n = (n+1)^2 + 2 \times \frac{1}{2}n(n+1) = (n+1)(n+1+n) = (n+1)(2n+1).$$

Thus the conjectured result holds for odd integers also.

As for the uniqueness, according to the results found, $10 = T_4 = 2^2 + T_2 + T_2 = 4 + 3 + 3$. However, an alternative expression is $10 = T_4 = 1 + 3 + 6 = 1^2 + T_2 + T_3$.

Remark: Note that the general result is stronger than stated: we can say that every triangular number greater than 1 is the sum of a square number and twice another triangular number.

2. A piece of wire 100 cm in length is bent into the shape of a sector of a circle. Find, in cm^2, the maximum value that the area A of the sector can have.

Solution

Let the radius of the sector be r and let its angle be θ. Denote the area of the sector by A cm^2. Since the length of the perimeter is 100 cm, the arc bounding the sector has length $100 - 2r$ cm. Further,

$$100 - 2r = \frac{\theta}{360}(2\pi r).$$

Similarly, in cm^2,

$$A = \frac{\theta}{360}\pi r^2.$$

From these two equations, it follows that

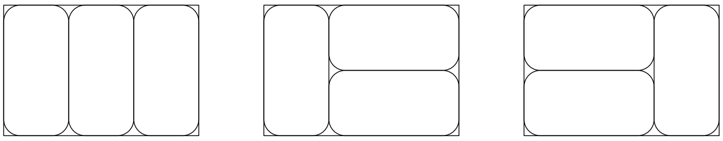
$$\frac{A}{100-2r}=\frac{r}{2} \quad \text{and hence} \quad A=(50-r)r.$$

On completing the square we have $A=(50-r)r=50r-r^2=25^2-(25-r)^2$. Hence, as $(25-r)^2 \geqslant 0$, it follows that $A \leqslant 25^2$. For a maximum, we need to check that there is a value of r for which $A=25^2=625$ and we see that this is $r=25$.

Remark 1: If you are familiar with the concept, then of course using radians makes the algebra in the question simpler.
Remark 2: You will observe that the sum of the arc length and twice the radius is a constant (100 cm). In general, when the sum of two numbers is constant, their product attains its maximum when the numbers are equal.

3. How many ways are there to tile a 2×15 grid with 2×1 dominoes?

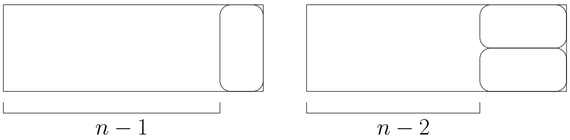
For example, there are three ways to tile a 2×3 grid with 2×1 dominoes:



Solution

Let $T(n)$ be the number of possible tilings of a $2 \times n$ grid. After trying out some small examples we find that $T(1)=1, T(2)=2, T(3)=3, T(4)=5$ and $T(5)=8$. This pattern should seem familiar: it is the start of the Fibonacci sequence, where each term is the sum of the two previous terms. This suggests that we want to prove that $T(n)=T(n-1)+T(n-2)$.

A tiling of a $2 \times n$ grid must end in either a vertical domino or in two horizontal dominoes.



In the first case, we are left with a $2 \times (n-1)$ grid, and so there are $T(n-1)$ possible tilings. In the second case, we are left with a $2 \times (n-2)$ grid, and so there are $T(n-2)$ possible tilings. Thus the total number of tilings of a $2 \times n$ grid is $T(n)=T(n-1)+T(n-2)$.

We can now use this to calculate all of the values of $T(n)$ up to $n=15$. We find that $T(15)=987$.

4. The integers 1, 2, 3, …, n are written on the board. Stephen picks any two of these numbers, a and b say, and erases them. He replaces them by $|a-b|$. He repeats this operation until just one integer remains. For which values of n is it possible for this last number to be 0?
Remark: Recall that $|a-b|$ is the absolute value or modulus of $a-b$.

Solution

The given process has appeared before, in Question 3 of Sheet 3, although in that question it was applied to a sum of integers. However, the two situations are not all that different: when the sequence is reduced to a single integer, that single integer is also the sum of the reduced sequence. This suggests that we might involve the sums of Stephen's numbers.

The first part of Sheet 3, Question 3 shows that, if we apply the process to ***sums*** of integers, then at each stage we are subtracting an ***even integer.*** In other words the parity of the original sum remains unchanged at each stage. Now 0 is even, so we have a chance of reaching 0 only if the initial sum is even. This does not provide a precise method of identifying the n's we seek so it is time to examine some small cases. It turns out that 0 can be obtained from 1, 2, 3 and 1, 2, 3, 4 but not from 1 and 1, 2. It also looks as though 1, 2, 3, 4, 5 will not reduce to 0. These observations lead to the following conjectures.

- 0 cannot be reached if $n = 4k + 1$ or $n = 4k + 2$, where k is a non-negative integer.
- 0 can be reached if $n = 4k$ or $n = 4k + 3$, where k is a non-negative integer.

(Note that, strictly speaking, k needs to be positive in the $n = 4k$ case.)

Now $T_n = 1 + 2 + \ldots + n = \frac{1}{2}n(n + 1)$. So when n is even, say $n = 2k$, we have $T_{2k} = k(2k + 1)$. Thus T_{2k} is even precisely when k is even, namely when n is is a multiple of 4. Treating n odd similarly: $T_{2k+1} = \frac{1}{2}(2k + 1)(2k + 2) = (2k + 1)(k + 1)$. In this case T_{2k+1} is even precisely when $k + 1$ is even, that is, when n leaves remainder 3 on division by 4. (Make sure that you understand why this is so, starting from, say, $k + 1 = 2m$.)

The foregoing shows that, for it to be possible for the last number to be 0, it is necessary that n leave remainder 0 or 3 on division by 4. It remains only to show that, in these cases, it is always possible for Stephen to end up with 0. If $n = 4k$, then the sequence can be divided up into subsequences which are each of the form $4j + 1, 4j + 2, 4j + 3, 4j + 4$, where j is a non-negative integer. Such a subsequence can be reduced as follows:

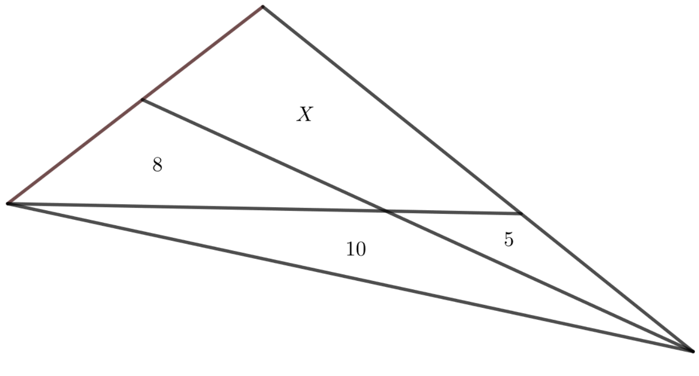
$$4j + 1,\ 4j + 2,\ 4j + 3,\ 4j + 4 \quad \longrightarrow \quad 1,\ 4j + 3,\ 4j + 4 \quad \longrightarrow \quad 1,\ 1 \quad \longrightarrow \quad 0.$$

Therefore the sequence can be reduced overall to a sequence of 0's; then such a sequence can be reduced to a single 0.

If $n = 4k + 3$, the sequence after 1, 2, 3 can be divided up into subsequences of length 4 as before. Thus the sequence can be reduced, eventually, to 1, 2, 3, 0. The final reduction is:

$$1,\ 2,\ 3,\ 0 \quad \longrightarrow \quad 2,\ 2,\ 0 \quad \longrightarrow \quad 0,\ 0 \quad \longrightarrow \quad 0.$$

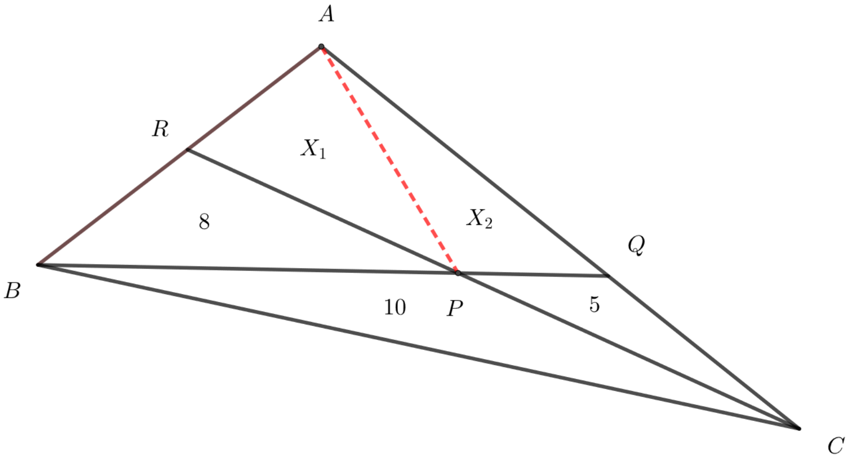
5. In the triangle shown below the three numbers denote areas.



Find the area X of the quadrilateral indicated.

SOLUTION

The triangle is shown again below with the addition of labelling and the line segment AP. The segment AP has been added to produce two more triangles. This will make it easier to compare unknown with known areas. The argument turns on the simple fact that if two triangles have the same height, then their areas are proportional to their bases. The tricky parts of the argument involve the selection of helpful triangles. In the solution the standard square brackets notation for an area is used: for example the area of a triangle UVW is usually denoted by $[UVW]$.



Consider the triangles ARC and ARP as having bases lying along the line RC and the same height, given by the perpendicular from A onto RC. These triangles involve the areas X_1 and X_2. Furthermore, on the other side of RC are triangles with known areas. Hence

$$\frac{[ARC]}{[ARP]} = \frac{RC}{RP} = \frac{[BCR]}{[BPR]}. \tag{1}$$

Making appropriate substitutions, (1) becomes

$$\frac{X_1 + (X_2 + 5)}{X_1} = \frac{RC}{RP} = \frac{18}{8} = \frac{9}{4}. \tag{2}$$

Now apply a similar argument to triangles based along the line QB.

$$\frac{[ABQ]}{[APQ]} = \frac{QB}{QP} = \frac{[CQB]}{[CQP]}. \tag{3}$$

Substituting once more, (3) yields

$$\frac{(8 + X_1) + X_2}{X_2} = \frac{QB}{QP} = \frac{15}{5} = 3. \tag{4}$$

Equations (2) and (4) furnish a pair of simultaneous equations in X_1 and X_2 and reduce to

$$5X_1 - 4X_2 = 20 \qquad \text{and} \qquad -X_1 + 2X_2 = 8.$$

From the second equation we have $X_1 = 2X_2 - 8$. Then, substituting in the first, $6X_2 = 60$; hence $X_2 = 10$. It follows that $X_1 = 12$ and so $X = X_1 + X_2 = 22$.

6. The function $x \mapsto f(x)$ satisfies the ***functional equation***

$$f(x+y) = f(x)f(y), \tag{*}$$

where x and y are real numbers.

a) Explain why, if there is a real number a such that $f(a) = 0$, then the function f is identically zero.

b) Now suppose that there is no real a such that $f(a) = 0$. Explain why $f(x) > 0$ for all real x and why $f(0) = 1$.

c) Prove that $f(nx) = (f(x))^n$ for all real numbers x and all integers n.

d) Given that $f(1) = 27$, find $f\left(\frac{2}{3}\right)$. Find $f(q)$ where q is a rational number.

SOLUTION

a) Suppose u is any real number. In (*) set $x = u - a$ and $y = a$ to obtain

$$f(u) = f((u-a)+a) = f(u-a)f(a) = 0.$$

Hence f is identically zero.

b) Equation (*) gives

$$f(x) = f\left(\tfrac{x}{2} + \tfrac{x}{2}\right) = \left[f\left(\tfrac{x}{2}\right)\right]^2 > 0.$$

Put $y = 0$ in (*); this gives $f(x) = f(x+0) = f(x)f(0)$. Then $f(x) \neq 0$ may be cancelled to obtain $f(0) = 1$.

c) First we deal with positive n. Setting $x = y$ in (*) gives $f(2x) = f(x+x) = f(x)f(x) = (f(x))^2$. Then $f(3x) = f(2x+x) = f(2x)f(x) = (f(x))^2 f(x) = (f(x))^3$ and so on. The argument can be completed with induction: if $f(nx) = (f(x))^n$ holds for all real x and a fixed positive integer n then $f((n+1)x) = f(nx+x) = f(nx)f(x) = f(x)^{n+1}$ so the result holds for $n+1$ as well.

Now for $n = 0$ we have $f(0) = 1 = f(x)^0$ which is what we want.

For negative n we can use what we know about positive n: let $n = -m$ where m is a positive integer. We have already shown $f(mx) = f(x)^m$. Now

$$1 = f(0) = f(mx+nx) = f(mx)f(nx) = f(x)^m f(nx),$$

so $f(nx) = f(x)^{-m} = f(x)^n$ as required.

d) Let $x = \frac{m}{n}$, where m and n are integers (and $n \neq 0$). Then $nx = m \times 1$ and it follows by applying part *c)* twice that

$$(f(x))^n = f(nx) = f(m \times 1) = (f(1))^m = 27^m.$$

$(f(x))^n = 27^m$ implies that $f(x) = 27^{\frac{m}{n}} = 27^x$. In particular $f\left(\frac{2}{3}\right) = 27^{\frac{2}{3}} = 9$; in general, for rational q, we have $f(q) = (f(1))^q$.

Remark: You may well suspect that, given $f(1) = 27$, then $f(x) = 27^x$ for all real numbers x. However, surprisingly this may not be the case! For those interested: assuming the axiom of choice there are very pathological solutions to the functional equation (that cannot be written down explicitly). To move from rational x to real x requires an extra condition on f such as it being continuous or increasing.

7. (a) In the quadrilateral $ABCD$, the sides AB and DC are parallel and the diagonal BD bisects $\measuredangle ABC$. Let X be the point of intersection of the diagonals AC and BD.

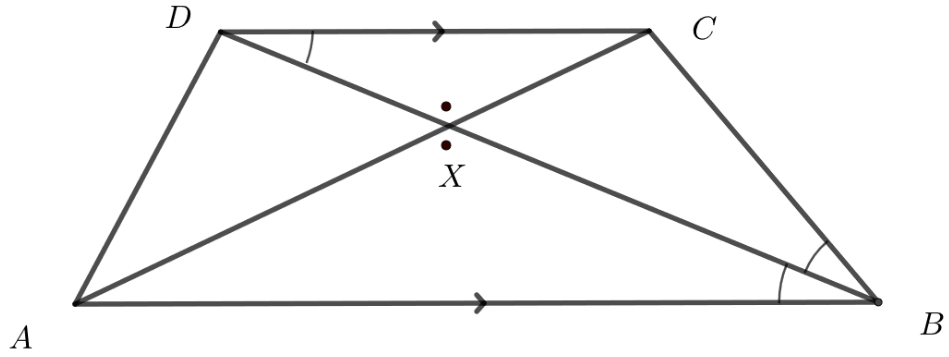
Prove that $\dfrac{AX}{XC} = \dfrac{AB}{BC}$.

(b) In triangle PQR, the lengths of all three sides are positive integers. The point M lies on the side QR so that PM is the internal bisector of $\measuredangle QPR$. Also $QM = 2$ and $MR = 3$. What are the possible lengths of the sides of triangle QPR?

[Mathematical Olympiad for Girls 2014 Question 4]

Solution

(a) The configuration is shown in the diagram below.



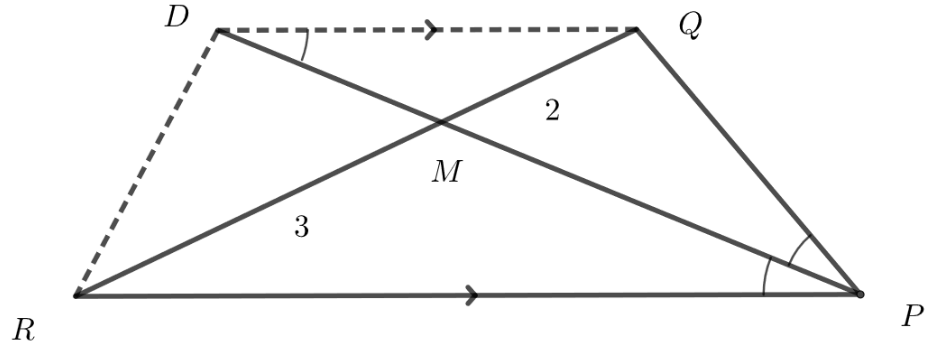
The angles at B are marked as equal because BD is given to be the internal bisector of $\measuredangle ABC$. Also, because the sides AB and DC are parallel, $\measuredangle ABD = \measuredangle BDC$ (alternate angles). Furthermore $\measuredangle AXB = \measuredangle DXC$ (vertically opposite angles). It immediately follows that triangles ABX and CDX are similar. In turn this implies that

$$\frac{AB}{CD} = \frac{BX}{DX} = \frac{AX}{XC}.$$

Now note that triangle BCD is isosceles because $\measuredangle DBC = \measuredangle BDC$. Hence $BC = CD$. Combining this with the equal ratios above yields $\dfrac{AX}{XC} = \dfrac{AB}{BC}$, as required.

Remark: This result for triangle ABC is called the Angle Bisector Theorem.

(b) This second diagram shows the given information about the triangle PQR. It has also been extended to illustrate the connection with the result in part (a).



That part (a) result yields $\dfrac{RP}{PQ} = \dfrac{RM}{MQ} = \dfrac{3}{2}$ or equivalently

$$RP = \frac{3}{2}PQ. \tag{1}$$

However, we also know that there are basic relations between between the sides of a triangle; these are enshrined in the triangle inequalities. The relevant inequalities follow with the substitutions provided by (1)

$$RP + PQ > QR \quad \text{or} \quad \frac{5}{2}PQ > 5, \tag{2}$$

$$PQ + QR > RP \quad \text{or} \quad PQ + 5 > \frac{3}{2}PQ, \tag{3}$$

$$QR + RP > PQ \quad \text{or} \quad 5 + \frac{3}{2}PQ > PQ. \tag{4}$$

Note first that (4) is always true and so contributes no useful information. However, (2) implies that $PQ > 2$. Further (3) shows that $PQ < 10$. Returning to (1), we also see that PQ must be even because all the sides of the triangle have integer length. Thus the possible lengths for PQ are 4, 6 and 8. Consequently the three cases are:

- $PQ = 4$, $QR = 5$ and $RP = 6$;
- $PQ = 6$, $QR = 5$ and $RP = 9$;
- $PQ = 8$, $QR = 5$ and $RP = 12$.

8. We are given a positive integer r and a chess board with 12 rows and 20 columns where each square has side length 1. You can move from a square to another one only if the distance between the centres of the two squares is $\sqrt{r}$ (so, for example, if $r = 2$ then you can only move between squares that are diagonally adjacent). The task is find a sequence of moves leading from the bottom left square to the bottom right square that stays within the chess board.

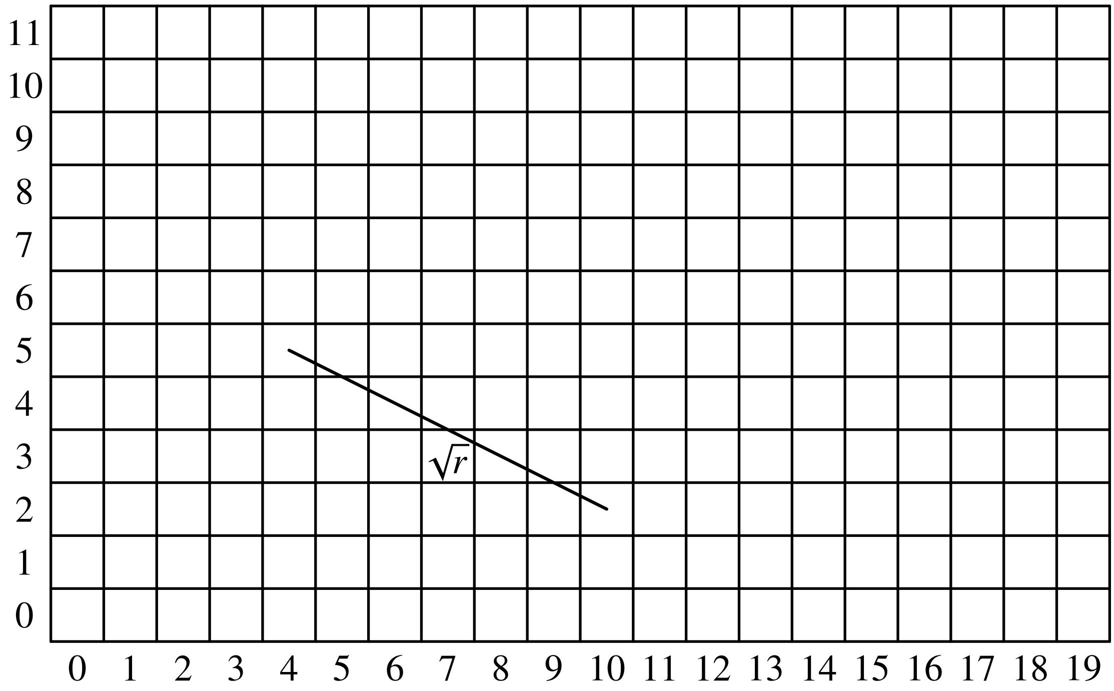
a) Show that the task cannot be done if r is divisible by 2 or 3.

b) Prove the task is possible when $r = 73$.

c) Can the task be done when $r = 97$?

Solution

To set up some notation we will assume that the centres of the squares have coordinates (x, y) for $x = 0, 1, \ldots, 19$ and $y = 0, 1, \ldots 11$ (so the centre of the bottom left square is $(0, 0)$, the centre of the bottom right square is $(19, 0)$ and the centre of the top right square is $(19, 11)$). We can move between (x, y) and $(x + a, y + b)$ only if $a^2 + b^2 = r$. Finally the task is to find a sequence of moves from $(0, 0)$ to $(19, 0)$.



a) First suppose r is even: $a^2 + b^2 = r$ so either a and b are both even or a and b are both odd. So we have two types of move: an even move (where we jump an even amount in both the x and y directions) or an odd move (where we jump an odd amount in both the x and y directions).

Suppose it is possible to find a sequence of moves from $(0, 0)$ to $(19, 0)$ and suppose that such a sequence uses E even moves and O odd moves.

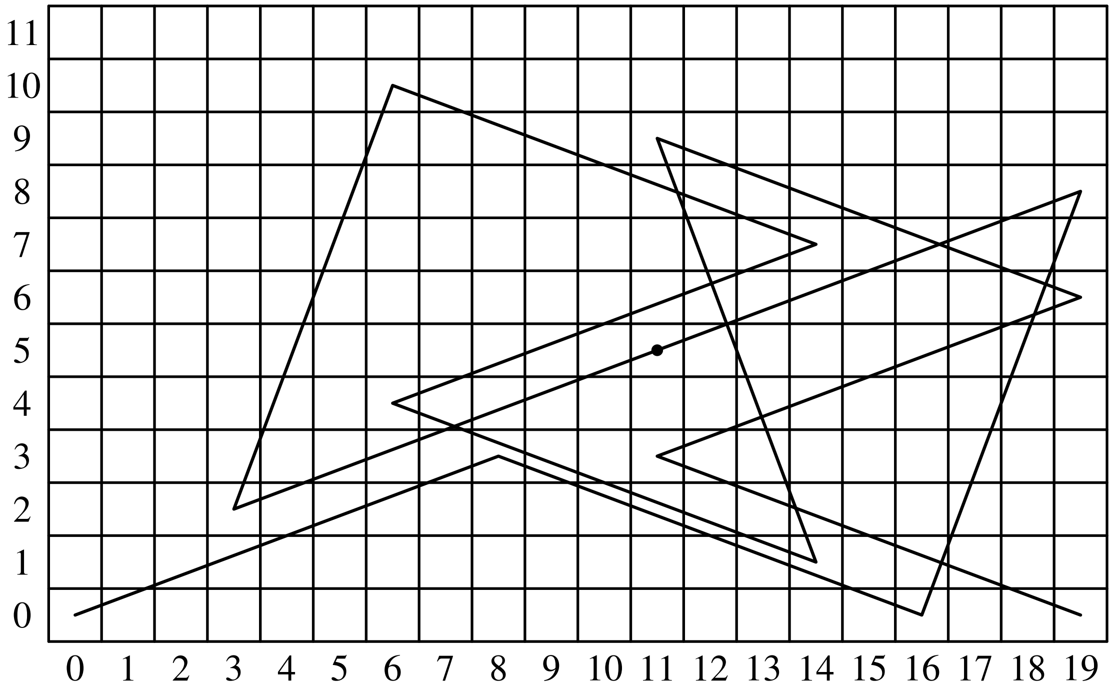
Overall we move an odd amount (19) in the x-direction so O must be odd. On the other hand we move an even amount (0) in the y-direction so O must be even. This is impossible.

Alternatively: A slick colouring argument also finishes off the problem: colour the board black and white chessboard-style. If we make an even move or an odd move then the colour of the two squares we have moved between is the same. However $(0, 0)$ and $(19, 0)$ are in different coloured squares so we cannot possibly find a sequence of moves from one to the other.

Now suppose r is a multiple of 3: $a^2 + b^2 = r \equiv 0 \bmod 3$. The squares modulo 3 are $0^2, 1^2, 2^2$ which are 0 or 1 modulo 3. As $a^2 + b^2 \equiv 0 \bmod 3$ we must have that both a^2 and b^2 are 0 modulo 3 so a and b are both divisible by 3. Hence all moves change the x-coordinate by a multiple of 3 so any sequence of moves changes the x-coordinate by a multiple of 3. Thus if we start at $(0, 0)$ then all squares we can get to much have x-coordinate divisible by 3. However 19 is not divisible by 3 so we cannot move between $(0, 0)$ and $(19, 0)$.

b) Note that $73 = 64 + 9 = 8^2 + 3^2$ so we can move from (x, y) to $(x \pm 8, y \pm 3)$ or to $(x \pm 3, y \pm 8)$. The following sequence of moves goes from $(0, 0)$ to $(19, 0)$:

$$(0,0) \to (3,8) \to (16,0) \to (19,8) \to (11,5) \to (3,2) \to (6,10) \to (14,7)$$
$$\to (6,4) \to (14,1) \to (11,9) \to (19,6) \to (11,3) \to (19,0)$$



c) The only way to write 97 as a sum of two squares is $97 = 81 + 16 = 9^2 + 4^2$ so the possible moves are

$$(x, y) \to (x \pm 4, y \pm 9),$$
$$(x, y) \to (x \pm 9, y \pm 4),$$

Call the first lot type 1 moves and the second lot type 2 moves. Type 1 moves change the x-coordinate by an even amount while type 2 moves change the x-coordinate by an odd amount. Suppose it were possible to move from $(0, 0)$ to $(19, 0)$ by a sequence of type 1 and type 2 moves. Since, overall we would have moved 19 in the x-direction we must have used an odd number of type 2 moves.

Having played around with various sequences of moves it seems that the board is not really big enough to let us move from $(0, 0)$ to $(19, 0)$: the moves seem to be too big. Any sequence starting at $(0, 0)$ which uses an odd number of type 2 moves always seems to end in the middle of the board.

Thinking about this further we see that type 1 moves always take us from the bottom four rows to the top four rows or vice versa while type 2 moves take us from the bottom four rows to the middle four rows (or vice versa) or from the middle four rows to the top four rows (or vice versa).

Let us call the the top four and bottom four rows the extreme rows and the middle four rows the middle rows. Then a type 1 move takes us between extreme rows while a type 2 move takes us from an extreme row to a middle row or from a middle row to an extreme row.

We have said that any sequence of moves from $(0, 0)$ to $(19, 0)$ uses an odd number of type 2 moves and starts in an extreme row (the bottom row) so must end in a middle row. But $(19, 0)$ is not in a middle row so this is impossible.

Remark: This was the first problem of the 1996 International Mathematical Olympiad held in Mumbai, India!